EXOTIC SPHERES

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Abstract. The goal of this thesis is to compute the size of the group Θ_n of h-cobordism classes of homotopy n-spheres for $n \geq 4$. We study Θ_n by studying the Kervaire-Milnor long exact sequence. Various constructions of exotic spheres are also discussed.

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1. Introduction

The goal of this thesis is to compute the size of the group Θ_n of h-cobordism classes of homotopy n-spheres, for $n \geq 4$ following [13]. For $n \geq 5$, the h-cobordism theorem implies that the group Θ_n is the same as the set of all diffeomorphism classes of differentiable structures on the topological n-sphere. A differentiable manifold that is homeomorphic, but not diffeomorphic, to S^n is called an exotic sphere. If $n \geq 5$ we can therefore think of elements of Θ_n as exotic spheres. The existence of exotic spheres was first proven by Milnor in [21]. In [21], Milnor constructed a class of exotic 7-spheres as 3-sphere bundles over the 4-sphere, proving that Θ_7 is nontrivial. For $n = 1, 2$, the group $\Theta_n = 0$. The main result of [13] which we discuss here is that Θ_n is finite for $n \geq 4$. We will study Θ_n by studying the Kervaire-Milnor long exact sequence

$$\cdots \to \Omega_{n+1}^{alm} \to P_{n+1} \to \Theta_n \to \Omega_n^{alm} \to P_n \to \cdots$$

which will be introduced in §5. Informally, Ω_n^{alm} is the group of cobordism classes of almost stably framed n-manifolds and P_{n+1} is the group of cobordism classes of parallelizable manifolds whose boundary are homotopy spheres. The map $\Theta_n \to \Omega_n^{alm}$ is essentially inclusion and the map $P_{n+1} \to \Theta_n$ is the boundary map. In §2, we give more precise definitions of these groups and maps.

The Kervaire-Milnor exact sequence is a useful tool for studying Θ_n since the groups Ω_n^{alm} and P_{n+1} are better understood. The study of Ω_n^{alm} is closely related to the J-homomorphism and the stable homotopy groups of spheres through the exact sequence

$$\cdots \to \pi_n SO \to \Omega_n^{fr} \to \Omega_n^{alm} \to \pi_{n-1} SO \to \cdots \tag{1}$$

Here Ω_n^{fr} is the group of cobordism classes of stably framed n-manifolds. Under the Ponytragin-Thom isomorphism, the map $\pi_n SO \to \Omega_n^{fr}$ above is the J-homomorphism. We will study the exact sequence (1) in §3. Using the work of Adams on the image of the J-homomorphism, these two exact sequences allow us to get upper-bounds for the image of Θ_n in Ω_n^{alm}. The image of P_{n+1} in Θ_n is studied using surgery. The exact order of Θ_n is then obtained by constructing elements of P_{n+1} using plumbing.

Behind the scenes in all of these computations is the fact that homotopy spheres are stably parallelizable. All of §3 is devoted to proving this fact.

We overview the contents of each section of this thesis.

Section 2: We give precise definitions of the groups Θ_n, Ω_n^{alm}, and P_{n+1} involved in the Kervaire-Milnor long exact sequence. The maps between these groups are also defined.

Section 3: It is shown that every homotopy sphere is stably parallelizable (Theorem 3.1). Let Σ be a homotopy n-sphere and ϵ^1 denote the trivial line bundle on Σ. The transition function for $T\Sigma \oplus \epsilon^1$ is then an element $\mathfrak{o}_n(\Sigma) \in \pi_{n-1}SO(n+1)$. Using Bott periodicity, we can compute the group $\pi_{n-1}SO(n+1)$. Since $\pi_{n-1}SO(n+1) = 0$ for $n \equiv 3, 5, 6, 7 \mod 8$, the obstruction $\mathfrak{o}_n(\Sigma)$ vanishes automatically. The proof of Theorem 3.1 then splits into two cases: the case when $n \equiv 1, 2 \mod 8$ and the case when $n \equiv 0, 4 \mod 8$ depending on whether $\pi_{n-1}SO(n+1)$ is $\mathbb{Z}/2$ or $\mathbb{Z}$.

In §3.2, the case $\pi_{n-1}SO(n+1) = \mathbb{Z}/2$ is addressed. By work of Adams [1], the J-homomorphism is injective in these dimensions. We therefore reduce our problem to showing $J(\mathfrak{o}_n(\Sigma)) = 0$ in π_*^S. The computation of $J(\mathfrak{o}_n(\Sigma))$ requires the second long exact sequence described above (1). The exactness of (1) is therefore proven in §3.2.

In §3.3, a separate argument is given for the case $\pi_{n-1}SO(n+1) = \mathbb{Z}$. In this case, the J-homomorphism is not injective so the argument given in §3.2 does not apply. Instead, we use an identification of the obstruction class $\mathfrak{o}_n(\Sigma)$ with the Pontryagin class $p_{n/4}(\Sigma)$. We then make use of the Hirzebruch signature theorem together with a computation of Bott of the first unstable homotopy group of $U(n)$.

Section 4: We define the quotient group Θ_n/bP_{n+1} using the definitions of §2 and argue that this quotient group is finite (Theorem 4.1). The result is essentially an application of the results of §3 together with Serre's theorem that the stable homotopy groups of spheres are finite ([27]). Proving that Θ_n is finite is therefore reduced to showing that the groups bP_{n+1} are finite. This is done in §§7-8. The proof of Theorem 4.1 will imply exactness of a sequence

$$P_{n+1} \longrightarrow \Theta_n \longrightarrow \mathrm{coker}(J_n) \tag{2}$$

Section 5: The Kervaire-Milnor long exact sequence is shown to be exact. The main technical difficulty here is translating from the exact sequence (2) described in §4 to the Kervaire-Milnor long exact sequence by replacing $\mathrm{coker}(J_n)$ with Ω_n^{alm}. We conclude §5 by stating the results of sections 6-9 in terms of the Kervaire-Milnor long exact sequence.

Section 6: The tools needed for §§7-8 are introduced. The theory used is referred to as "surgery." The main idea is to attach handles to a manifold in order to kill its homotopy groups. Kervaire and Milnor's paper [13] can be viewed as the introduction of surgery theory for simply connected manifolds. Since [13], surgery theory has been greatly extended and studied. See for instance Browder's book [7] and Lück's notes [16]. Although we will not discuss it here, there is a more general "surgery exact sequence" which is the Kervaire-Milnor long exact sequence for spheres. In §6 we prove various results on surgery in the middle dimension as well as discuss when a surgery can be framed. It is shown that framed surgery is a framed cobordism invariant so that we can perform framed surgery on a representative of an element of P_{n+1} without changing its cobordism class in P_{n+1}.

Section 7: It is shown that the groups P_{2k+1} are zero. This is done by performing surgery on an element of P_{2k+1} until all of its homotopy groups are killed. The case k even is addressed first in §7.1. The case k odd requires more care in terms of framing the surgery. This is done in §7.2.

Section 8: The groups P_{2k} are shown to be finite. Let $[M]$ be a class in P_{2k}. First we show that the only obstruction to killing all the homotopy of M using surgery is the existence of a weakly symplectic basis $\{\lambda_i\}$ for H_kM so that each λ_i can be represented by an embedded sphere with trivial normal bundle (c.f. Lemma 8.1). The proof then breaks into the cases of k even and k odd.

For k even, trivializing normal bundles of embedded spheres is not an issue (Lemma 8.3). The existence of a weakly symplectic basis for H_kM is dictated by the signature of M. The signature is used to define an injective map $\mathrm{sign} : P_{2k} \to \mathbb{Z}$. The map sign is shown to be surjective using a "plumbing" construction to create manifolds with a specified signature (Theorem 8.4). The image of $b : \mathbb{Z} \cong P_{2k} \to \Theta_{2k-1}$ is then determined using the Kervaire-Milnor long exact sequence. The order of bP_{2k} is computed in terms of the order of the image of the J-homomorphism. Using [1], we can compute the exact value of $|bP_{2k}|$.

The case of k odd is treated in §8.2. Since k is odd, symplectic bases of H_kM always exist. The barrier to applying Lemma 8.1 is the triviality of the normal bundle of embedded spheres. The information of the triviality of these normal bundles can be packaged in the form of the Kervaire invariant. The Kervaire invariant then defines a map $c : P_{2k} \to \mathbb{Z}/2$. Using a plumbing construction, this map is seen to be an isomorphism. The Kervaire-Milnor exact sequence then implies that the image of $b : P_{2k} \to \Theta_{2k-1}$ is either 0 or $\mathbb{Z}/2$, depending on whether or not there exists an almost stably framed $(2k+1)$-manifold with Kervaire invariant one. The solution to the Kervaire invariant one problem by Hill, Hopkins, and Ravenel is then used to compute bP_{2k} for $2k \neq 126$.

Section 9: Here we collect the results proven in the above sections. The order of Θ_n is fully described in terms of the order of $\mathrm{coker}(J_n)$. Moreover, the group Θ_n is described up to a group extension problem that was solved by Brumfiel in [8] and [9].

Section 10: We apply the results summarized in §9 to give explicit values of the group Θ_n for $n \leq 9$.

Section 11: An explicit construction on the exotic 8-sphere is given. In §8 and §9 the elements of bP_n are described using a plumbing construction. The exotic 8-sphere is the first instance of an exotic sphere not bounding a parallelizable manifold. In §11 we describe a generalization of the plumbing construction called the Milnor pairing. The pairing is a map

$$\pi_{n-1}SO(m) \times \pi_{m-1}SO(n) \to \Theta_{n+m-1}$$

for $n < m$. We show that the exotic 8-sphere is in the image of this pairing. The proof that the sphere constructed using the Milnor pairing is in fact exotic uses a result of Smith [28]. In [28], the Milnor pairing is studied by viewing certain pairs $(\alpha, \beta) \in \pi_{n-1}SO(m) \times \pi_{m-1}SO(n)$ as a rank m vector bundle ξ_α on S^n together with a stable framing on $\mathsf{Sph}(\xi_\alpha)$ that is twisted by β. The corresponding stably framed cobordism class $[\mathsf{Sph}(\xi_\alpha); \beta] \in \Omega^{fr}_{n+m-1}$ can be computed in certain cases. It is shown that the exotic 8-sphere corresponds to the sphere bundle $SU(3)$ on S^5 together with its left-invariant framing $\mathcal{L}$. We cite a computation by Smith, Steer and Wood of $[SU(3); \mathcal{L}]$ in the cokernel of J to complete the proof ([28], [29], [33]).

Section 12: An appendix is included. The appendix contains a discussion of the proof of Bott periodicity, an introduction to characteristic classes, a proof of a result used in §3.3 relating characteristic classes to obstruction classes, and a proof of the Hirzebruch signature theorem. The appendix also includes a brief overview of the major results stated in this note without proof.

We assume the reader is familiar with various tools in algebraic topology including characteristic classes, classifying spaces, homology, and stable homotopy groups. Additionally, we will use, without proof, the following theorems: Pontryagin-Thom, Serre's finiteness of π_r^S, Adams' work on the image of J along with the proof of the Adams' conjecture, Bott's computation of the first unstable homotopy group of the unitary groups, Hill-Hopkins-Ravenel's solution of the Kervaire invariant problem, computation of the rth stable stems for small r, the identification of $SU(3)$ with its Lie group framing as an element of π_8^S, and the computation of the oriented cobordism ring over $\mathbb{Q}$. See the Appendix for more discussion.

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2. Definitions

Throughout this note, manifold will mean a compact, oriented, smooth manifold. The point of this section is to give precise definitions of the groups and maps in the Kervaire-Milnor long exact sequence

$$\cdots \to \Omega^{alm}_{n+1} \xrightarrow{R} P_{n+1} \xrightarrow{b} \Theta_n \xrightarrow{i} \Omega^{alm}_n \xrightarrow{R} P_n \to \cdots$$

2.0.1. *The group* Θ_n.

Definition. A homotopy n-sphere is a closed manifold that is homotopy equivalent to S^n.

For oriented M, let $-M$ denote M with the reversed orientation.

Definition. We say that two n-manifolds M and M' are h-cobordant if $M \sqcup -M'$ bounds an $(n+1)$-manifold W such that M and M' are deformation retracts of W.

Let Θ_n be the set of homotopy n-spheres modulo h-cobordism.

Claim. *Under connect sum, Θ_n is a group. The unit is the standard sphere and inverses are given by reversing orientation.*

For a proof see [13, Thm. 1.1]. In the course of the proof of the claim, the following lemma is proven. Since we will need it later, we include it here.

Lemma 2.1. *A homotopy sphere $\Sigma \in \Theta_n$ bounds a contractible manifold if and only if Σ is h-cobordant to the standard sphere.*

Proof. If W is an h-cobordism between Σ and S^n, then attaching a disk along S^n, we obtain a manifold W' with $\partial W' = \Sigma$. Since W deformation retracts onto S^n, the new manifold W' is contractible.

Conversely, say $\Sigma = \partial W'$ with W' contractible. Remove an open disk from W' to obtain a simply connected manifold W with $\partial W = \Sigma \sqcup (-S^n)$. We need W to deformation retract onto Σ and S^n. Since Σ and S^n are simply connected, it suffices to show that the inclusions $S^n \to W$ and $\Sigma \to W$ induce isomorphisms on homology. We have a commutative diagram from the long exact sequences of the pairs $(\mathbb{D}^{n+1}, S^n)$ and (W', W),

$$\begin{tikzcd}
0 \arrow[r] & H_{i+1}(\mathbb{D}^{n+1}, S^n) \arrow[r, "\cong"] \arrow[d, "\cong"'] & H_i S^n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_{i+1}(W', W) \arrow[r, "\cong"'] & H_i W \arrow[r] & 0
\end{tikzcd}$$

Thus the inclusion induced map $H_i S^n \to H_i W$ is an isomorphism.

For the inclusion $\Sigma \subset W$, note that Poincaré duality gives an isomorphism $H_i(W, \Sigma) \cong H^{n+1-i}(W, S^n)$. Since W deformation retracts onto S^n, the inclusion $\Sigma \subset W$ must also induce an isomorphism on homology. Since Σ is simply connected, this implies that W deformation retracts onto Σ. $\square$

2.0.2. *The group Ω_n^{alm}.* An almost stable framing of an (oriented) pointed manifold (M, x) is a trivialization of $(TM \oplus \epsilon^1)|_{M \setminus \{x\}}$ that gives $M \setminus \{x\}$ the orientation that agrees with the restriction of the orientation on M. Two almost stably framed n-manifolds $(M, x, \bar{u})$ and $(N, y, \bar{v})$ are almost stably framed cobordant if there exists a triple (W, γ, ψ) so that

(i) W is an $(n+1)$-manifold with $\partial W = M \sqcup -N$,

(ii) $\gamma : [0,1] \to W$ is an embedding with $\gamma(0) = x$ and $\gamma(1) = y$, and

(iii) ψ is a trivialization of $(TW \oplus \epsilon^1)|_{W \setminus \operatorname{Im} \gamma}$ that restricts to $\bar{u}$ and $\bar{v}$ on M and N, respectively.

The almost stably framed cobordism classes of almost stably framed n-manifolds forms a group Ω_n^{alm} under connect sum. Define a map $i : \Theta_n \to \Omega_n^{alm}$ by sending $\Sigma \in \Theta_n$ to the almost stably framed manifold Σ with framing on $(T\Sigma \oplus \epsilon^1)|_{\Sigma \setminus \{x\}}$ given by contracting $\Sigma \setminus \{x\}$.

Lemma 2.2. *The map $i : \Theta_n \to \Omega_n^{alm}$ is a well-defined group homomorphism.*

Proof. Say Σ_0, Σ_1 are h-cobordant homotopy spheres. Let W be an h-cobordism between Σ_0 and Σ_1. Pick any path $\gamma : [0,1] \to W$ between $x \in \Sigma_0$ and $y \in \Sigma_1$. Since W deformation retracts onto Σ_0, $W \setminus \operatorname{Im} \gamma$ is contractible. The trivialization of $(TW \oplus \epsilon^1)|_{W \setminus \operatorname{Im} \gamma}$ given by contraction restricts to the trivializations on $i(\Sigma_0)$ and $i(\Sigma_1)$. Thus i is well-defined.

Since the group operation on both groups is connect sum, it is clear that i is a group map. $\square$

2.0.3. *The group P_n.* Let $\tilde{P}_n$ be the collection of parallelizable n-manifolds $(M, \bar{u})$ whose boundary is a homotopy sphere. Define $(M, \bar{u})$ and $(N, \bar{v})$ to be framed cobordant if there exists a parallelizable $(n+1)$-manifold W so that

(i) the boundary $\partial W = M \cup_{\partial M} H \cup_{\partial N} N$ where H is an h-cobordism between ∂M and ∂N, and

(ii) the trivialization of TW restricts to the trivializations $\bar{u}$ and $\bar{v}$ on TM and TN, respectively. Define P_n to be the group of framed-cobordism classes of elements of $\tilde{P}_n$.

We think of the identity element in P_n as a disk, rather than a point. The group structure on P_n is connect sum along the boundary. See [13, pg. 507] for details.

Let $b : P_n \to \Theta_{n-1}$ be the boundary map. By our definitions, b is clearly a well-defined group map. Define a map $R : \Omega_n^{alm} \to P_n$ by sending $(M, x, \bar{u}) \in \Omega_n^{alm}$ to the manifold M' obtained by removing a small open disk around x from M. Now $\bar{u}$ gives a trivialization of $TM' \oplus \epsilon^1$. Since M' is a manifold with boundary, Corollary 3.3 below implies that M' is parallelizable.

Lemma 2.3. *The map $R : \Omega_n^{alm} \to P_n$ is a well-defined group map.*

Proof. Say (W, γ, ψ) is an almost stably framed cobordism between $(M, x, \bar{u})$ and $(N, y, \bar{v}) \in \Omega_n^{alm}$. Removing a tubular neighborhood ν of $\operatorname{Im}\gamma$ from W results in a manifold W' with boundary

$$\partial W' = R(M) \cup_{S^n} \partial\nu \cup_{S^n} R(N)$$

Now ν, being a bundle over the contractible space $\operatorname{Im}\gamma$, is trivial. Thus $\partial\nu = [0,1] \times S^n$ is an h-cobordism between $\partial R(M)$ and $\partial R(N)$. Hence R is well-defined. □

3. Stable Parallelizability

The point of this section is to give a detailed proof of the following theorem.

Theorem 3.1. *Every homotopy sphere is stably parallelizable.*

Definition. A manifold M is stably parallelizable if the Whitney sum $TM \oplus \epsilon^1$ of the tangent bundle of M with a trivial line bundle is trivial.

Throughout, let Σ be a homotopy n-sphere. We will prove Theorem 3.1 by showing that the obstruction class for $T\Sigma \oplus \epsilon^1$ being trivial always vanishes. Our first step is therefore to describe the obstruction class.

3.1. **Obstruction Construction.**

Since Σ is a homotopy sphere, the vector bundle $T\Sigma \oplus \epsilon^1$ is determined by its clutching function $\mathfrak{o}_n(\Sigma) \in \pi_{n-1}(SO(n+1))$. We can describe the clutching function in two ways.

First, fix any $x \in \Sigma$ and let $S^{n-1} \subset \Sigma$ be a small sphere around x. Now S^{n-1} splits Σ into two contractible "hemispheres,"

$$\Sigma = D^n \cup (\Sigma \setminus \{x\})$$

where $x \in D^n \subset \Sigma$ is the disk bounded by S^{n-1}. The contraction of D^n determine a unique (up to isotopy) trivialization of $T\Sigma \oplus \epsilon^1$ restricted to D^n. Restricting this trivialization to $S^{n-1} \subset D^n$, we get a trivialization of $(T\Sigma \oplus \epsilon^1)|_{S^{n-1}}$. Similarly, the contraction of $\Sigma \setminus \{x\}$ determines a trivialization of $(T\Sigma \oplus \epsilon^1)|_{S^{n-1}}$. Comparing these two trivializations gives a map $\mathfrak{o}_n(\Sigma) \in \pi_{n-1}(SO(n+1))$. Note that this is also the obstruction to extending the trivialization on $\Sigma \setminus \{x\}$ to all of Σ.

Remark. The above discussion works for any manifold $M \in \Omega_n^{alm}$, defining a map $\partial : \Omega_n^{alm} \to \pi_{n-1}SO(n+1)$ sending $M \mapsto \mathfrak{o}_n(M)$. Showing homotopy spheres are stably parallelizable is then the same as showing the composition

$$\Theta_n \xrightarrow{i} \Omega_n^{alm} \xrightarrow{\partial} \pi_{n-1}SO(n+1)$$

is zero.

Second, we can use classifying maps to get $\mathfrak{o}_n(\Sigma)$. Let $f : \Sigma \to BSO(n+1)$ classify $T\Sigma \oplus \epsilon^1$. Then f defines a class

$$[f] \in [\Sigma, BSO(n+1)] = \pi_n(BSO(n+1)) \cong \pi_{n-1}(SO(n+1))$$

The image of $[f]$ in $\pi_{n-1}(SO(n+1))$ is $\mathfrak{o}_n(\Sigma)$.

Remark. For a general manifold M, the obstruction class should live in $H^n(M;\pi_{n-1}SO(n+1))$. Since Σ is a homotopy n-sphere,

$$H^n(\Sigma;\pi_{n-1}SO(n+1)) \cong \pi_{n-1}(SO(n+1))$$

Indeed, if $K(\pi,n) = K(\pi_{n-1}(SO(n+1)),n)$ is an Eilenberg-MacLane space, then

$$H^n(\Sigma,\pi_{n-1}(SO(n+1))) = [\Sigma, K(\pi,n)] = [S^n, K(\pi,n)] = \pi_n(K(\pi,n)) = \pi_{n-1}(SO(n+1)) \tag{3}$$

We discuss this more general construction in the Appendix (12.11).

Our obstruction class lives in $\pi_{n-1}(SO(n+1))$. Since $\pi_{n-1}SO(n+1)$ is in the stable range, we will consider $\mathfrak{o}_n(\Sigma)$ as living in $\pi_{n-1}(SO)$. By Bott periodicity [3],

$n \mod 8$	0	1	2	3	4	5	6	7
$\pi_{n-1}(SO)$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0

Thus when $n \equiv 3,5,6,7 \mod 8$, we have $\mathfrak{o}_n(\Sigma) \in \pi_{n-1}(SO) = 0$ and there is nothing to show. The rest of the proof is broken into two cases, depending on $n \mod 8$: when $\pi_{n-1}(SO) = \mathbb{Z}$ and when $\pi_{n-1}(SO) = \mathbb{Z}/2$.

3.2. **Case I.** When $n \equiv 1,2 \mod 8$, we have $\mathfrak{o}_n(\Sigma) \in \pi_{n-1}(SO) = \mathbb{Z}/2$. The proof that $\mathfrak{o}_n(\Sigma) = 0$ will go as follows:

Step 1: Show that $J(\mathfrak{o}_n(\Sigma)) = 0$.
Step 2: By Adams' work on the J-homomorphism, J is injective for $n \equiv 1,2 \mod 8$.

Lemma 3.2. *Let ξ be a rank k oriented vector bundle on an n-manifold M with $k > n$. Then ξ is trivial if and only if ξ is stably trivial. If M has nontrivial boundary, the same is true for $k = n$.*

Proof. It suffices to consider the case when $\xi \oplus \epsilon^1$ is trivial. Let $f : M \to BSO(k)$ classify ξ. We have a fiber bundle

$$S^k \to BSO(k) \xrightarrow{\pi} BSO(k+1)$$

where π is induced by inclusion. The vector bundle classified by $\pi \circ f : M \to BSO(k+1)$ is $\xi \oplus \epsilon^1$. By assumption, we have a nullhomotopy F between f and the constant map at some $x_0 \in BSO(k+1)$,

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] & BSO(k) \arrow[d, "\pi"] \\ M \times I \arrow[r, "F"] \arrow[ur, dashed, "H"] & BSO(k+1) \end{tikzcd}$$

By the homotopy lifting property, we get a homotopy H between $H_0 = f$ and $H_1 : M \to \pi^{-1}(x_0) \simeq S^k$. If $k > n = \dim(M)$ or if $k = n$ and M has nontrivial boundary, the map H_1 must be nullhomotopic. Thus f is nullhomotopic and hence ξ is trivial. □

Corollary 3.3. *Let W be a manifold with nonempty boundary. Then TW is stably trivial if and only if it is trivial.*

Corollary 3.4. *Let M be an oriented n-manifold. Let $i : M \to \mathbb{R}^{n+k}$ be an embedding with normal bundle $\nu(i)$. If $\nu(i)$ is trivial, then $TM \oplus \epsilon^k$ is trivial. If $k > n$, the converse also holds.*

Proof. Say $\nu(i) \cong \epsilon^k$. Then $TM \oplus \epsilon^k \cong TM \oplus \nu(i) \cong \epsilon^{n+k}$. Conversely, say $TM \oplus \underline{\mathbb{R}}^k$ is trivial. Then

$$\nu(i) \oplus \epsilon^{n+k} \cong \nu(i) \oplus TM \oplus \epsilon^k \cong \epsilon^{n+k} \oplus \epsilon^k$$

is trivial. Since $\nu(i)$ has rank $k > n$, Lemma 3.2 implies that $\nu(i)$ is trivial. □

Remark. We could instead say "the tangent bundle is stably trivial if and only if the stable normal bundle is trivial." The proof then follows from the fact that in $[M, BSO]$, we have $[TM] = -[\nu]$.

We will start by proving something that is true for all n, but only helpful for us when $n \equiv 1, 2 \mod 8$.

Let Ω_{n-1}^{fr} be cobordism classes of $(n-1)$-manifolds together with a stable framing of their tangent bundle. Define $\bar{J} : \pi_{n-1}(SO) \to \Omega_{n-1}^{fr}$ by sending $\gamma : S^{n-1} \to SO$ to the cobordism class of S^{n-1} with framing given by twisting the standard stable framing of S^{n-1} by γ.

Lemma 3.5. *The composition*

$$\Theta_n \xrightarrow{i} \Omega_n^{alm} \xrightarrow{\partial} \pi_{n-1}(SO) \xrightarrow{\bar{J}} \Omega_{n-1}^{fr}$$

is zero.

Proof. Let $(M, x, \bar{u}) \in \Omega_n^{alm}$ where $\bar{u}$ is a trivialization of $TM \oplus \epsilon^1$ restricted to $M \setminus \{x\}$. To show that $\bar{J}\partial(M)$ defines a trivial cobordism class, we need an n-manifold with boundary S^{n-1} and framing that restricts to $\partial(M)$ on S^{n-1}. But the framing given by $\partial(M)$ is defined to be the restriction of the framing given by $\bar{u}$ on $M \setminus \{x\}$. Removing the open disk containing x and bounded by $S^{n-1} \subset M$, we obtain an n-manifold with boundary giving a framed cobordism between $\bar{J}\partial(M)$ and the empty set. $\square$

In fact

$$\Omega_n^{alm} \xrightarrow{\partial} \pi_{n-1}(SO) \xrightarrow{\bar{J}} \Omega_{n-1}^{fr}$$

is an exact sequence of abelian groups. Moreover, it can be extended to a long exact sequence:

Lemma 3.6. *Let $f : \Omega_n^{fr} \to \Omega_n^{alm}$ be the forgetful map. Then*

$$\cdots \to \pi_n(SO) \xrightarrow{\bar{J}} \Omega_n^{fr} \xrightarrow{f} \Omega_n^{alm} \xrightarrow{\partial} \pi_{n-1}(SO) \xrightarrow{\bar{J}} \Omega_{n-1}^{fr} \xrightarrow{f} \Omega_{n-1}^{alm} \to \cdots$$

is exact.

Since we will need this long exact sequence later on, we prove its exactness here.

Proof. By Lemma 3.5, we have $\bar{J}\partial = 0$. We start by showing $\ker(\bar{J}) \subset \operatorname{Im} \partial$. Let $\alpha \in \pi_{n-1}SO$ with $\bar{J}\alpha = 0$. Then there exists a framed n-manifold M with $\partial M = S^{n-1}$ and a stable framing $\bar{u}$ of M restricting to α on ∂M. Consider the manifold $Q = M \cup_{S^{n-1}} \mathbb{D}^n$ obtained by capping off M. After removing a point from $x \in \mathbb{D}^n$ the space $Q \setminus \{x\}$ retracts onto M and therefore is framed. Now the obstruction to extending $\bar{u}$ to all of Q is α. Thus $\partial(Q) = \alpha$.

Before checking that the sequence is exact at Ω_n^{alm} and Ω_n^{fr}, we need to check that f is a group map. Let $M, M' \in \Omega_n^{fr}$. To show that f is a group map, we need to prove that $M \# M'$ and $M \sqcup M'$ are almost stably framed cobordant. The manifold $M \# M'$ is obtained from $M \sqcup M'$ by surgery on an element in $\pi_0(M \sqcup M')$. Thus $M \# M'$ and $M \sqcup M'$ are cobordant. By Lemma 6.4 below, this surgery, and hence the cobordism, can be framed.

Next we show exactness at Ω_n^{alm}. Let $(M, \bar{u}) \in \Omega_n^{fr}$. Then $\partial f(M)$ is the obstruction to extending the framing $\bar{u}|_{M \setminus \{x\}}$ to all of M. Since $\bar{u}$ is defined on all of M, $\partial \circ f(M) = 0$. Conversely, let $(N, \bar{u}, x) \in \Omega_n^{alm}$ with $\partial(N) = 0$. Then $\bar{u}$ extends to all of N and hence $N \in \operatorname{Im}(f)$.

Finally, we show exactness at Ω_{n-1}^{fr}. Let $\alpha \in \pi_{n-1}(SO)$. Then $f\bar{J}(\alpha)$ is the manifold S^{n-1} with framing on $S^{n-1} \setminus \{x\}$ given by α. Now S^{n-1} bounds the disk $\mathbb{D}^n$ and the framing on $S^{n-1} \setminus \{x\}$ agrees with the framing on $\mathbb{D}^n$ given by contractibility. Thus $f\bar{J}(\alpha) = 0$. Note that $\bar{J}(\alpha) \in \Omega_{n-1}^{fr}$ is not necessarily zero since the weird framing on S^{n-1} given by α does not need to extend to the standard framing on $\mathbb{D}^n$. Conversely, let $(Q, \bar{v}, x) \in \Omega_{n-1}^{fr}$ with $f(Q) = 0$. Then there exists

- an n-manifold N with $\partial N = Q$,
- an embedding $\gamma : [0, 1] \to N$ with $\gamma(0) = x$,

- and a trivialization $\bar{u}|_{N\setminus \operatorname{Im}\gamma}$ so that $\bar{u}|_{\partial N\setminus\{x\}} = \bar{v}|_{Q\setminus\{x\}}$.
Take a collar neighborhood U of Q in N. Then $\bar{u} = \bar{v}$ extends to U. We thus have a framing $\bar{v}$ on $N \setminus \gamma(\epsilon, 1]$ for some small ϵ. Removing a small tubular neighborhood ν of $(\epsilon, 1] \subset N$ from N, we obtain a framed manifold N' with $\partial N' = Q \sqcup \partial\nu$. Since ν is a trivial bundle, $\partial\nu = S^n$. Thus N' is a framed cobordism between Q and a sphere. □

The composition of $\bar{J}$ with the Pontryagin-Thom construction is the standard J-homomorphism.

Definition. The J-homomorphism is defined as follows. Given $\gamma \in \pi_{n-1}(SO(n+1))$, we can act on $\mathbb{R}^{n+1}$ by γ. Taking one-point compactifications, γ defines a pointed map $S^{n+1} \to S^{n+1}$. Now J_{n-1} is the composition

$$\pi_{n-1}(SO(n+1)) \to [S^{n-1}, Maps_+(S^{n+1}, S^{n+1})] \cong [S^{n-1} \wedge S^{n+1}, S^{n+1}] \cong [S^{n-1+n+1}, S^{n+1}] = \pi_{2n}(S^{n+1})$$

Since everything works well with suspensions, we get a stable map $J : \pi_{n-1}SO \to \pi_{n-1}^S$.

For details on the Pontryagin-Thom construction, see [31].

Lemma 3.7. *The following diagram commutes:*

$$\begin{tikzcd}
 & \Omega_{n-1}^{fr} \arrow[dd, "\phi"] \\
\pi_{n-1}(SO) \arrow[ur, "\bar{J}"] \arrow[dr, "J"'] & \\
 & \pi_{n-1}^S
\end{tikzcd}$$

Here π_{n-1}^S is the $(n-1)$ stable homotopy group of the sphere spectrum.

Proof. Let $\gamma : S^{n-1} \to SO(n+1)$. Then $\bar{J}(\gamma)$ is the cobordism class of S^{n-1} with framing given by γ and $J(\gamma) : S^{2n} \to S^{n+1}$ defines a stable homotopy class in $\pi_{2n-n+1}^S = \pi_{n-1}^S$. The inverse of the Pontryagin-Thom map $\phi^{-1} : \pi_{n-1}^S \to \Omega_{n-1}^{fr}$ sends a homotopy class $\alpha : S^{2n} \to S^{n+1}$ to the framed manifold N obtained by taking the preimage $N := \alpha^{-1}\{y\}$ of a regular value y.The framing on $\alpha^{-1}\{y\}$ is given by pulling back a basis of $T_y S^{n+1}$ along $d_x\alpha$ to a basis of $T_x S^{2n} = T_x N \oplus \nu_x = \nu_x$. Since y is a regular value, this gives an isomorphism $\nu_x \cong T_y S^{n+1}$ for each $x \in N$, i.e., a framing of the normal bundle of N. (Recall that by Lemma 3.2, this is equivalent to a stable framing of TN).

To show that $\phi^{-1}(J\gamma) = \bar{J}(\gamma)$, we will first show that $(J\gamma)^{-1}\{y\}$ is a sphere for y a regular value. The way we defined $J\gamma$ gives us an identification of S^{2n} as $S^{n-1} \wedge (\mathbb{R}^{n+1})^+$. Using this identification, take y to be any point not involving the point at infinity. Now

$$J\gamma : S^{n-1} \wedge (\mathbb{R}^{n+1})^+ \to S^{n+1}$$

sends (a, b) to $\gamma_a(b)$. Since $\gamma_a \in SO(n+1)$ is invertible, there is a unique $b \in (\mathbb{R}^{n+1})^+$ so that $\gamma_a(b) = y$. Thus $(J\gamma)^{-1}(y) = S^{n-1}$.

We still need to show that the framing on S^{n-1} given by γ agrees with the framing for $\phi^{-1}(J\gamma)$. This is basically the observation that $J\gamma : S^{2n} \to S^{n-1}$ comes from a map $S^{n-1} \times S^{n+1} \to S^{n+1}$ sending $(u, v) \mapsto \gamma(u) \cdot (v)$. But this is exactly the map defining the framing $\bar{J}(\gamma)$. □

Everything we have said so far works for any n: we always have $J(\mathfrak{o}_n(\Sigma)) = 0$. To see that $\mathfrak{o}_n(\Sigma) = 0$, we want the J-homomorphism to be injective. It is a very deep theorem of Adams that this is true for $n \equiv 1, 2 \mod 8$.

Theorem 3.8 (Adams [1])**.** *For $n \equiv 1, 2 \mod 8$, the J-homomorphism $J : \pi_{n-1}(SO) \to \pi^S_{n-1}$ is injective.*

Corollary 3.9. *For $n \equiv 1, 2 \mod 8$, the obstruction $\mathfrak{o}_n(\Sigma)$ to Σ being stably parallelizable is zero. In fact, $\partial = 0$ when $n \not\equiv 0 \mod 4$.*

For $n \equiv 0, 4 \mod 8$, we have $\pi_{n-1}SO = \mathbb{Z}$ by Bott periodicity. Since the stable homotopy groups of spheres are finite, there is no hope of J being injective. We therefore need a different proof for homotopy spheres of dimension $n = 4k$.

Remark. Note that $\mathfrak{o}_n(\Sigma) : S^{n-1} \to SO(n+1)$ is nullhomotopic if and only if it extends to a map from a disk

$$\begin{tikzcd} \mathbb{D}^n \arrow[dr] & \\ S^{n-1} \arrow[u] \arrow[r, "\mathfrak{o}_n(\Sigma)"'] & SO(n+1) \end{tikzcd}$$

From the above, $J\mathfrak{o}_n(\Sigma)$ being zero implies that we can extend the map $\mathfrak{o}_n(\Sigma)$ to the disk. The way we proved $J\mathfrak{o}_n(\Sigma) = 0$ was by noting that the framing $S^{n-1} \to SO(n+1)$ on the equator extended to the southern hemisphere of Σ. Extending to the southern hemisphere is *not* the same as extending the map $\mathfrak{o}_n(\Sigma)$ to the disk as above. Extending $\mathfrak{o}_n(\Sigma)$ to a disk is the same as extending the trivialization to the northern hemisphere, not the southern hemisphere.

3.3. **Case II.** Say $n = 4k$. The obstruction is then an integer, $\mathfrak{o}_n(\Sigma) \in \pi_{n-1}(SO) \cong \mathbb{Z}$. The proof in this case will go as follows:

(1) Show that $\mathfrak{o}_n(\Sigma)$ is a (nonzero) multiple of the Pontryagin class $p_k(\Sigma)$.
- Interpret the Pontryagin class as an obstruction class. (This will be done in the Appendix.)
- Understand the homotopy groups of $SO(m), U(m)$ and the maps between them.

(2) Show that $\langle p_k(\Sigma), [\Sigma]\rangle$ is a (nonzero) multiple of the signature $\mathrm{sign}(\Sigma)$.
- This is Hirzebruch's signature theorem.

(3) Note that $\mathrm{sign}(\Sigma) = 0$.

This first two steps work more generally for elements of Ω^{alm}_n. The only part where we are using the fact that Σ is a homotopy sphere is in (3). Since we will need the higher level of generality in §6, we will work in Ω^{alm}_n here. Throughout, let $M \in \Omega^{alm}_{4k}$.

By $p_k(M)$ we mean the Pontryagin class associated to the vector bundle $TM \to M$. This is the same as the Pontryagin class associated with the vector bundle $TM \oplus \epsilon^1$.

Lemma 3.10. *Let a_k be 1 if k is even and 2 if k is odd. Then*

$$p_k(M) = \pm a_k(2k-1)!\mathfrak{o}_n(M)$$

Proof. By Lemma 12.13 below, $p_k(M) = h_*q_*\mathfrak{o}_n(M)$ where

$$\pi_{n-1}(SO(n+1)) \xrightarrow{q_*} \pi_{n-1}(U(n+1)) \xrightarrow{h_*} \pi_{n-1}\Big(U(n+1)/U(2k-1)\Big)$$

are induced by inclusion and quotienting. By Bott periodicity, all of these homotopy groups are $\mathbb{Z}$, so we can write

$$\mathbb{Z} \xrightarrow{q_*} \mathbb{Z} \xrightarrow{h_*} \mathbb{Z}$$

We will show that q_* is multiplication by $(2k-1)!$ and that h_* is multiplication by a_k. The $\pm$ sign in the statement of the lemma comes from possibly different choices of generators for $\mathbb{Z}$. We will then have

$$p_k(M) = h_* q_* \mathfrak{o}_n(M) = \pm a_k (2k-1)! \mathfrak{o}_n(M)$$

as claimed.

Note that h_* comes from the inclusion $SO(n+1) \subset U(n+1)$. Consider the following part of the long exact sequence for the pair $(SO(n+1), U(n+1))$

$$\pi_{n-1}(SO(n)) \xrightarrow{h_*} \pi_{n-1}(U(n+1)) \to \pi_{n-1}(U(n)/SO(n)) \to \pi_{n-2}(SO(n+1))$$

By Bott periodicity, this looks like

$$\mathbb{Z} \xrightarrow{h_*} \mathbb{Z} \to \mathbb{Z}/a_k\mathbb{Z} \to 0$$

Thus $h_*(1) = \pm a_m$.

For q_*, consider the long exact sequence for the pair $(U(2k-1), U(n+1))$:

$$\pi_{n-1}(U(n+1)) \xrightarrow{q_*} \pi_{n-1}(U(n+1), U(2m-1)) \to \pi_{n-2}U(2m-1) \to \pi_{n-2}U(n)$$

Note that $\pi_{n-2}(U(2k-1))$ is of the form $\pi_{2m}(U(m))$. By [4, Thm. 5], $\pi_{2m}(U(m)) = \mathbb{Z}/m!\mathbb{Z}$ (c.f. Appendix §12.5.4) Combining this result with Bott periodicity, we have an exact sequence

$$\mathbb{Z} \xrightarrow{q_*} \mathbb{Z} \to \mathbb{Z}/(2k-1)!\mathbb{Z} \to 0$$

Thus q_* is multiplication by $(2k-1)!$. □

Let

$$s_k = \frac{2^{2k}(2^{2k-1}-1)B_k}{(2k)!}$$

where B_k is the kth Bernoulli number.

Claim (Hirzebruch Signature Theorem)**.** *Let M a $4k$-manifold. Then*

$$\operatorname{sign}(M) = \langle L(M), [M] \rangle$$

where $L(M)$ is a certain polynomial in the variables $p_1(M), \ldots, p_k(M)$ so that

$$L(M) = s_k p_k(M) + \{\textit{terms not involving } p_k(M)\}$$

For a proof see the Appendix, Theorem 12.14.

Corollary 3.11. *For $(M, \bar{u}, x) \in \Omega_{4k}^{alm}$, we have* $\operatorname{sign}(M) = \langle s_k p_k(M), [M] \rangle$.

Proof. By the signature theorem, it suffices to show that $p_i(M) = 0$ for $i < k$. Since $M \setminus \{x\}$ is parallelizable, $p_i(M \setminus \{x\}) = 0$ for all i. The inclusion $M \setminus \{x\} \to M$ induces an isomorphism on cohomology in degrees less than $4k-2$. For $i < k$, we therefore have $p_i(M) = p_i(M \setminus \{x\}) = 0$. □

Theorem 3.12. *For $n = 0, 4 \mod 8$ the obstruction class $\mathfrak{o}_n(\Sigma)$ is zero.*

Proof. By Lemma 3.11, $\operatorname{sign}(\Sigma) = \langle s_k p_k(\Sigma), [\Sigma] \rangle$. But the signature is a bilinear form on $H^{2k}(\Sigma) = 0$, so the signature $\operatorname{sign}(\Sigma) = 0$. Since the pairing $\langle -, [\Sigma] \rangle$ is nondegenerate, $s_k p_k(\Sigma) = 0$. But then

$$0 = s_k p_k(\Sigma) = \pm s_k a_k (2k-1)! \mathfrak{o}_n(\Sigma)$$

Since $s_k a_k (2k-1)! \neq 0$, the obstruction class must vanish. □

This completes Case II and therefore the proof of Theorem 3.1.

4. Finiteness of Θ_n/bP_{n+1}

As a consequence of the results of §3, we will show that the quotient group Θ_n/bP_{n+1} is finite.

Definition. Define $bP_{n+1} \subset \Theta_n$ to be the set of homotopy spheres that bounded parallelizable manifolds.

Note that bP_{n+1} is the image of the map $b : P_{n+1} \to \Theta_n$ defined in §2.

Theorem 4.1. *The quotient group Θ_n/bP_{n+1} is finite.*

Proof. Recall from Lemma 2.2 that we have a map $i : \Theta_n \to \Omega_n^{alm}$ defined by sending $\Sigma \in \Theta_n$ to Σ with the framing of $(T\Sigma \oplus \epsilon^1)|_{\Sigma\setminus\{x\}}$ given by the contractibility of $\Sigma \setminus \{x\}$. But by Theorem 3.1, $T\Sigma \oplus \epsilon^1$ is trivial on all of Σ. Thus $\Theta_n \to \Omega_n^{alm}$ lands in the image of the forgetful map $f : \Omega_n^{fr} \to \Omega_n^{alm}$. By Lemma 3.6, $\mathrm{Im}(f) = \Omega_n^{fr}/\mathrm{Im}(\bar{J})$. By Lemma 3.7, $\Omega_n^{fr}/\mathrm{Im}(\bar{J}) \cong \mathrm{coker}(J_n)$. We therefore get a homomorphism $F : \Theta_n \to \mathrm{coker}(J_n)$. The kernel of F is exactly those homotopy spheres that, together with a stable framing of their tangent bundle, define trivial elements of Ω_n^{fr}. Thus Σ is in $\ker(F)$ if and only if Σ bounds a stably framed $(n+1)$-manifold. By Corollary 3.3, this is equivalent to bounding a parallelizable manifold. Hence $\ker(F) = bP_{n+1}$ and Θ_n/bP_{n+1} is a subgroup of $\mathrm{coker}(J_n)$. It is a theorem of Serre that the stable homotopy groups of spheres are finite. Thus Θ_n/bP_{n+1} is finite. □

5. Kervaire-Milnor Exact Sequence

This section is roughly following [15, Appendix]. Recall from §2.0.3 that the boundary map defines a group homomorphism $b : P_{n+1} \to \Theta_n$. The proof of Theorem 4.1 implies that

$$P_{n+1} \xrightarrow{b} \Theta_n \xrightarrow{F} \mathrm{coker}(J_n)$$

is exact, i.e., $\Theta_n/bP_{n+1} \cong \mathrm{Im}\, F \subset \mathrm{coker}(J_n)$. To compute this image, it would be nice to extend the above exact sequence to the right. We will define a map $r : \mathrm{coker}(J_n) \to P_n$ extending this exact sequence.

Define $r : \mathrm{coker}(J_n) \to P_n$ as follows. First define $\tilde{r} : \pi_n^S \to P_n$ to be the composition

$$\pi_n^S \xrightarrow{\simeq} \Omega_n^{fr} \to \Omega_n^{alm} \to P_n$$

where

- the first map is the Pontryagin-Thom isomorphsim,
- the second map is the forgetful map, and
- the third map takes an almost parallelizable manifold $(M, x, \bar{u})$ to the parallelizable manifold with boundary obtained by removing a disk around x from M.

Let $\gamma \in \pi_n(SO)$. The image of γ in Ω_n^{alm} is the (standard) sphere S^n with framing on $S^n \setminus \{pt\}$ given by the contraction. Removing a disk from the sphere, we obtain a contractible manifold, so $\tilde{r}(\gamma) \in P_n$ is the cobordism class of a disk.Thus $\tilde{r}$ descends to a map $r : \mathrm{coker}(J_n) \to P_n$.

Proposition 5.1. *The sequence*

$$P_{n+1} \xrightarrow{b} \Theta_n \xrightarrow{F} \mathrm{coker}(J_n) \xrightarrow{r} P_n$$

is exact.

Proof. We only need to show exactness at $\mathrm{coker}(J_n)$. Let $\Sigma \in \Theta_n$. Embed $\Sigma \subset \mathbb{R}^{n+k}$ with $k > n$. Let ϕ be a framing of the normal bundle of Σ. Then $F(\Sigma)$ is the coset in $\pi_n^S/\mathrm{Im}(J_n)$ corresponding (under the Pontryagin-Thom isomorphism) to the framed manifold $(\Sigma, \phi) \in \Omega_n^{fr}$. Thus $rF(\Sigma)$ is the cobordism class in P_n of Σ minus a disk. But removing a disk from Σ leaves us with something contractible. Thus $rF = 0$.

Now let $\alpha \in \pi_n^S$ so that $r([\alpha]) = 0$. Let (M, ϕ) be the framed manifold that α corresponds to under the Pontryagin-Thom isomorphism. Then removing a disk from M, we obtain a manifold $M \setminus D_1 \in P_n$ that is framed cobordant to a disk. Let W be a framed cobordism between $M \setminus D_1$ and a disk D_2. Then the manifolds $(M \setminus D_1) \cup_{S^{n-1}} D_1$ and $(D_2 \cup_{S^{n-1}} D_1)$ are framed cobordant, i.e., in Ω_n^{fr} we have

$$[(M \setminus D_1) \cup_{S^{n-1}} D_1] = [(D_2 \cup_{S^{n-1}} D_1)]$$

But $D_2 \cup_{S^{n-1}} D_1$ is a homotopy sphere. Thus $\alpha \in \operatorname{Im}(F)$. □

This will help us understand Θ_n/bP_{n+1} but is not as helpful in understanding bP_{n+1}. Ideally, we could continue extending this exact sequence to a long exact sequence.

Theorem 5.2. *If $n \not\equiv 0 \mod 4$, then*

$$P_{n+1} \to \Theta_n \to \operatorname{coker}(J_n) \to P_n \to \Theta_{n-1} \to \operatorname{coker}(J_{n-1}) \to P_{n-1}$$

is exact.

The problem with extending this to a long exact sequence is that for $n = 4k$, the kernel of $P_n \to \Theta_{n-1}$ is larger than $\operatorname{coker}(J_n)$. To fix this, we need to replace $\operatorname{coker}(J_n)$ with the larger group

$$\pi_n^S \cong \Omega_n^{fr} \to \Omega_n^{alm}$$

of almost stably framed n-manifolds. As a consequence of Lemma 3.6 we have

Lemma 5.3. *The kernel of $\pi_n^S \to \Omega_n^{alm}$ is exactly $\operatorname{Im}(J_n)$ so that we have an injection*

$$j : \operatorname{coker}(J_n) \to \Omega_n^{alm}$$

If $n \not\equiv 0 \mod 4$, this injection is an isomorphism: $\operatorname{coker}(J_n) \cong \Omega_n^{alm}$.

When $n \equiv 0 \mod 4$, the map $\partial : \Omega_n^{alm} \to \mathbb{Z}$ is a multiple of the signature of the manifold, which has a possibility of being nonzero (cf. Lemma 8.5).

Let $i : \Theta_n \to \Omega_n^{alm}$ be the inclusion and $R : \Omega_n^{alm} \to P_{n+1}$ by removing a disk as in §2.

Corollary 5.4. *We have a commutative diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & \operatorname{coker}(J_{n+1}) \arrow[r, "r"] \arrow[d, "j"] & P_{n+1} \arrow[r, "b"] \arrow[d, equal] & \Theta_n \arrow[r, "F"] \arrow[d, equal] & \operatorname{coker}(J_n) \arrow[r] \arrow[d, "j"] & \cdots \\
\cdots \arrow[r] & \Omega_{n+1}^{alm} \arrow[r, "R"] & P_{n+1} \arrow[r, "b"] & \Theta_n \arrow[r, "i"] & \Omega_n^{alm} \arrow[r] & \cdots
\end{tikzcd}$$

Theorem 5.5 (Kervarie-Milnor Long Exact Sequence)**.** *We have a long exact sequence*

$$\cdots \to \Omega_{n+1}^{alm} \xrightarrow{R} P_{n+1} \xrightarrow{b} \Theta_n \xrightarrow{i} \Omega_n^{alm} \xrightarrow{R} P_n \to \cdots$$

Proof. By Corollary 5.4, injectivity of j, and exactness from Proposition 5.1, we have

$$\ker(i) = \ker(j \circ F) = \ker(F) = \operatorname{Im}(b)$$

and $R \circ i = r \circ F = 0$. The same proof that is given for Proposition 5.1 shows that $\ker(R) = \operatorname{Im}(i)$. We just need $\ker b = \operatorname{Im}(R)$. For $M \in \Omega_{n+1}^{alm}$, the element $R(M)$ is the manifold obtained by removing a disk from M. Then $bR(M)$ is the boundary of this removed disk, hence the standard sphere, $bR(M) = [S^n] = 0 \in \Theta_n$.

To see that $\ker(b) \subset \operatorname{Im}(R)$, say $(M, \phi) \in P_{n+1}$ is a framed $(n+1)$-manifold with boundary the standard sphere S^n. Then we can attach $\mathbb{D}^{n+1}$ to M along $S^n = \partial M$ to obtain a new $(n+1)$-manifold $\overline{M}$. The result of removing a point $x \in \mathbb{D}^{n+1}$ from $\bar{M}$, is a manifold that deformation retracts onto M. Since M is stably framed, $\overline{M}$ is almost stably framed. Thus $(\overline{M}, \phi) \in \Omega_{n+1}^{alm}$ and $R(\overline{M}, \phi) = (M, \phi)$. □

Remark. Let $M \in P_{n+1}$ with boundary an exotic sphere $\Sigma \in \Theta_n$. Then we can sill form the union $M \cup_\Sigma \mathbb{D}^{n+1}$ where S^n is identified with Σ through some homeomorphism. However, this union need not be a smooth manifold. For example, Kervaire's manifold with no differentiable structure is of this form (cf. [12] or Corollary 8.16 below).

Let $n \not\equiv 0 \mod 4$. Using the commutativity of the diagram in Corollary 5.4 and the equality $\operatorname{coker}(J_n) = \Omega_n^{alm}$ from Lemma 5.3, the Kervaire-Milnor long exact sequence reduces to the exact sequence of Theorem 5.2.

The rest of this note will involve computing the groups P_n and the maps $R : \Omega_n^{alm} \to P_n$. The results will be as follows:

Theorem 5.6. *The groups P_n only depend on n mod 4 with*

$$P_n = \begin{cases} 0 & n \text{ odd} \\ \mathbb{Z}/2 & n \equiv 2 \mod 4 \\ \mathbb{Z} & n \equiv 0 \mod 4 \end{cases}$$

The map $\Omega_{4k}^{alm} \to P_{4k} = \mathbb{Z}$ is $\frac{1}{8}$sign and the map $\Omega_{4k+2}^{alm} \to P_{4k+2} = \mathbb{Z}/2$ is the Kervaire invariant.

As a result we obtain,

Theorem 5.7. *The Kerviare-Milnor long exact sequence becomes,*

- *for $n \equiv 0 \mod 4$ with $n \geq 8$,*
$$0 \to \Theta_n \to \Omega_n^{alm} \xrightarrow{\frac{1}{8}sign} \mathbb{Z} \to bP_n \to 0$$
- *for $n \equiv 2 \mod 4$, $n \geq 6$,*
$$0 \to \Theta_n \to \Omega_n^{alm} \xrightarrow{c} \mathbb{Z}/2 \to bP_n \to 0$$
- *for $n \geq 5$ odd,*
$$0 \to bP_{n+1} \to \Theta_n \to \Omega_n^{alm} \to 0$$

We can use $\operatorname{coker}(J_n)$ rather than Ω_n^{alm} by Theorem 5.2 and the discussion after the proof of Theorem 5.5. Since elements of $\operatorname{coker}(J_n)$ come from stably framed manifolds $\Omega_n^{fr} \to \Omega_n^{alm}$, the map $\frac{1}{8}\text{sign} : \operatorname{coker}(J_n) \to \mathbb{Z}$ is zero.[1] The first claim in the theorem then states that for $n \equiv 0 \mod 4$ with $n \geq 8$, we have $\Theta_n \cong \operatorname{coker}(J_n)$.

6. Surgery Results

To compute the groups P_n, we will study the question of when a manifold can be transformed into a contractible manifold by a sequence of framed surgeries (cf. Lem 2.1). To do so, we need some standard results of surgery theory.

Definition. Let M be an n-manifold with $n = p + q + 1$. Let
$$\varphi : S^p \times \mathbb{D}^{q+1} \to M$$
be a smooth embedding. Form a new manifold
$$\chi(M, \varphi) = (M \setminus \varphi(S^p \times \{0\})) \cup (\mathbb{D}^{p+1} \times S^q)$$
where $\varphi(u, tv)$ is glued to (tu, v) for every $u \in S^p$, $v \in S^q$, $t \in (0, 1]$. Say $\chi(M, \varphi)$ is obtained from M by the surgery $\chi(\varphi)$. [2]

[1] Closed framed manifolds have trivial Pontryagin classes. By the Hirzebruch signature theorem, the signature of such manifolds are zero.

[2] In [13], the term "spherical modification" is used instead.

Note that the boundary of $\chi(M,\varphi)$ is the same as the boundary of M.

To use surgery theory to study P_n, we need to know that it is a cobordism invariant.

Lemma 6.1. *The manifolds M and $\chi(M,\varphi)$ are cobordant.*

Proof. Let $W = (M\times[0,1]) \cup (\mathbb{D}^{p+1}\times\mathbb{D}^{q+1})$ be the pushout

$$\begin{tikzcd} S^p\times\mathbb{D}^{q+1} \arrow[r] \arrow[d, "\varphi"] & \mathbb{D}^{p+1}\times\mathbb{D}^{q+1} \arrow[d] \\ M\times\{1\}\subset M\times I \arrow[r] & W \end{tikzcd}$$

The space W is referred to as the trace of the surgery $\chi(\varphi)$. Then $\partial W = M\sqcup\chi(M,\varphi)$. The only difficulty is smoothing out the corners of W. This is done in [19, Thm. 1]. $\square$

The following tells us the result of performing surgery below the middle dimension.

Lemma 6.2. *Let $\lambda\in\pi_pM$ denote the homotopy class of $\varphi|_{S^p\times\{0\}}$. Then the homotopy groups of $\chi(M,\varphi)$ are*

$$\pi_i(\chi(M,\varphi)) = \begin{cases}\pi_iM & i<\min(p,q)\\ \pi_pM/\Lambda & i=p\end{cases}$$

where Λ is a subgroup of π_pM containing λ.

In particular, when $p<q$ (i.e., $p\le(n-1)/2$), the surgery kills λ and leaves the lower homotopy groups the same.

Proof. Milnor gives a clear proof of this in [19, Lem 2]. We repeat it here.

Let W be the trace of $\chi(\varphi)$ and $M' = \chi(M,\varphi)$. Consider the subset $W_0 = M\cup(\mathbb{D}^{p+1}\times\{0\})$ of W. The space W_0 looks like M with a $(p+1)$-cell attached along the map $u\mapsto\varphi(u,0)$. Thus $\pi_i(M)\to\pi_i(W_0)$ is an isomorphism for $i<p$ and surjective for $i=p$ with λ in the kernel. Since W_0 is a deformation retract of W, we can say the same for $\pi_i(M)\to\pi_i(W)$. Similarly, if we consider M' embedded in W, we get isomorphism $\pi_i(M')\to\pi_i(W)$ for $i\le p$. The lemma follows. $\square$

To kill an element $\lambda\in\pi_pM$, we need λ to come from an embedding $\varphi: S^p\times\mathbb{D}^{n-p}\to M$.

Lemma 6.3. *If M^n is stably parallelizable and $p<n/2$, then λ is homotopic to the restriction of an embedding $\varphi: S^p\times\mathbb{D}^{n-p}\to M$.*

Proof. Since $n\ge 2p+1$, the Whitney embedding theorem implies that λ is homotopic to an embedding $\varphi_0: S^p\to M$ (cf. [32, Thm. 2]). Now $\varphi_0(S^p)$ has a stably trivial tangent bundle $T\varphi_0(S^p)\oplus\epsilon^1\cong\epsilon^{p+1}$. Let ν be the normal bundle of φ_0. Then

$$\epsilon^{n+1} = TM\oplus\epsilon^1 = \nu\oplus T\varphi_0(S^p)\oplus\epsilon^1 = \nu\oplus\epsilon^{p+1}$$

so that ν is stably trivial. Note that ν has rank $n-p$ which is bigger than the dimension of its base space S^p by assumption on p. By Lemma 3.2 ν is trivial. By the tubular neighborhood theorem, there is an embedding φ of ν in M making the diagram

$$\begin{tikzcd} S^p\times\mathbb{D}^{n-p} \arrow[d] \arrow[dr] & \\ S^p \arrow[r, "\varphi_0"'] & M \end{tikzcd}$$

commute. Thus λ is homotopic to $\varphi|_{S^p}$. $\square$

Let $M \in P_n$ with $n \geq 2p+1$. For any $\lambda \in \pi_p M$, we can construct $\chi(M, \varphi)$. Since our manifolds $(M, \bar{u}) \in P_n$ are framed, we want the new manifold $\chi(M, \varphi)$ to be framed as well. Let $W = (M \times [0,1]) \cup (\mathbb{D}^{p+1} \times \mathbb{D}^{q+1})$ be the trace of $\chi(\varphi)$.

Definition. A surgery $\chi(\varphi)$ is framed if there is a trivialization F of TW so that $F|_M = \bar{u}$.

If $\chi(\varphi)$ is framed, then $M' = \chi(M, \varphi)$ is stably parallelizable with trivialization given by $\bar{u}' = F|_{M'}$. Moreover, if $\chi(\varphi)$ is a framed, then M and $\chi(M, \varphi)$ are framed cobordant.

Proposition 6.4. *Let $(M, \bar{u})$ be a stably framed n-manifold with $n = p+q+1$. Assume $p \leq q$ and let $\lambda \in \pi_p M$. Then there exists an embedding $\varphi : S^p \times \mathbb{D}^{n-p} \to M$ so that $\chi(\varphi)$ is framed.*

Proof. By Lemma 6.3, there exists some embedding $\varphi : S^p \times \mathbb{D}^{n-p} \to M$ representing λ. Let W be the trace of $\chi(\varphi)$. The obstructions to extending the framing $\bar{u}$ to a framing on W live in the cohomology groups $H^{r+1}(W, M; \pi_r SO(n+1))$. Now W deformation retracts onto $W_0 = M \cup (\mathbb{D}^{p+1} \times \{0\})$. Thus $W/M_0 \simeq W_0/M$ looks like $\mathbb{D}^{p+1} \times \{0\}$ with the part glued to M killed. The part glued to M is $S^p \times \{0\}$ so that $W/M \simeq \mathbb{D}^{p+1}/S^p = S^{p+1}$. Hence our obstruction lives in

$$H^{r+1}(W, M; \pi_r SO(n+1)) = \begin{cases} \pi_p SO(n+1) & r = p \\ 0 & r \neq p \end{cases}$$

Let $\gamma(\varphi) \in \pi_p SO(n+1)$ the unique (possibly) nontrivial obstruction. The element $\gamma(\varphi)$ may be nonzero, but we can alter the embedding φ to obtain a new embedding φ_α with $\gamma(\varphi_\alpha) = 0$.

The alteration of φ is done as follows. Let $\alpha : S^p \to SO(q+1)$ and define

$$\varphi_\alpha : S^p \times \mathbb{D}^{q+1} \to M$$

by $\varphi_\alpha(u, v) = \varphi(u, \alpha(u)v)$. Since φ and φ_α agree on $S^p \times \{0\}$, they define the same homotopy class $\lambda \in \pi_p M$. Let $s_* : \pi_p SO(q+1) \to \pi_p SO(n+1)$ be the map induced by inclusion.

<u>Claim</u>: $\gamma(\varphi_\alpha) = \gamma(\varphi) + s_*(\alpha)$.

Since the inclusion $SO(q+1) \to SO(n+1)$ is q-connected, if $p \leq q$ then the map s_* is surjective. Choose α so that $s_*(\alpha) = -\gamma(\varphi)$. $\square$

Remark. The element $\gamma(\varphi) \in \pi_p SO(n+1)$ can be defined more geometrically as follows. It is the obstruction to extending the trivialization $\bar{u}$ on M to all of W. It suffices to extend $\bar{u}$ to $M \cup (\mathbb{D}^{p+1} \times \{0\})$. The trivialization $\bar{u}$ is already defined on M, so we just need to define a trivialization on the image of $\mathbb{D}^{p+1} \times \{0\}$ in W. Let $i : \mathbb{D}^{p+1} \times \{0\} \to W$ be the embedding. Now $i(S^p \times \{0\})$ is glued to $\varphi(S^p \times \{0\}) \subset M \times \{1\}$ in W, and we have $\bar{u}$ defined there. Thus $\gamma(\varphi)$ is the obstruction to extending $\bar{u}$ from $\varphi(S^p \times \{0\})$ to $i(\mathbb{D}^{p+1} \times \{0\})$. This extension is the same as a nullhomotopy of the map $\gamma(\varphi) : S^p \to SO(n+1)$ defining $\bar{u}$ on $\varphi(S^p \times \{0\})$.

Remark. The element α can be identified geometrically as well. Let ν be the normal bundle of $\varphi : S^p \times \{0\} \to M$. A trivialization of the rank $(q+1)$-bundle ν is given by a map $\alpha : S^p \to SO(q+1)$. The tubular neighborhood theorem then gives us an embedding $\varphi'_\alpha : S^p \times \mathbb{D}^{q+1} \to M$. Chasing through definitions, one sees that $\varphi'_\alpha = \varphi_\alpha$.

We still need to prove the claim from the lemma.

Proof of Claim. We want to show that $\gamma(\varphi_\alpha) = \gamma(\varphi) + s_*(\alpha)$. Let W and W_α be the traces of $\chi(\varphi)$ and $\chi(\varphi_\alpha)$. Then $\gamma(\varphi_\alpha) \in \pi_p SO(n+1)$ is the obstruction to extending $\bar{u}$ from $\varphi_\alpha(S^p \times \{0\})$ to $i_\alpha(\mathbb{D}^{p+1} \times \{0\})$. Let $t^{n+1} = e^{p+1} \times e^{q+1}$ the standard framing on $\mathbb{D}^{p+1} \times \mathbb{D}^{q+1}$. Then $\gamma(\varphi_\alpha)$ is the homotopy class of $g : S^p \to SO(n+1)$ by $g(u) = di_\alpha(t^{n+1}) \cdot \bar{u}$. On $S^p \times \{0\}$ we have

$$di_\alpha(t^{n+1}) = di(e^{p+1}) \times d\varphi_\alpha(e^{q+1})$$

At $(u, 0) \in S^p \times \mathbb{D}^{p+1}$, the derivative

$$d_{(u,0)}\varphi_\alpha(e^{q+1}) = \alpha(u) \cdot d_0\varphi(e^{q+1})$$

so

$$d_{(u,0)}i_\alpha(t^{n+1}) = s(\alpha) \cdot d_{(u,0)}i(t^{n+1})$$

Thus

$$di_\alpha(t^{n+1})\bar{u} = s(\alpha) \cdot di(t^{n+1})\bar{u}$$

□

The choose of α so that φ_α is framed is not unique. For example, when $p = q$, the map s_* has kernel $\mathbb{Z}$. We will need this freedom in the choice of α later (cf. Prop. 7.5).

Combining everything we have shown so far, we have the following.

Corollary 6.5. *Every element in P_n, $n \geq 2k$ is represented by some $(k-1)$ connected manifold.*

If we could kill the middle homotopy group as well, our manifold would be contractible.

Lemma 6.6. *Let M be a closed, oriented, manifold of dimension $2k$ or $2k+1$. If M is k-connected, then M is contractible.*

Proof. We will prove this for $n = 2k+1$, a similar argument works for $n = 2k$. By Hurewicz's theorem, $H_i(M) = 0$ for $i \leq k$ and $H_{k+1}(M) = \pi_{k+1}(M)$. By Poincaré duality, $H^j(M) = 0$ for $j \geq k+1$. By the universal coefficients theorem, $\mathrm{Tor}(H_j(M), \mathbb{Z}) = 0$ and $\mathrm{Hom}(H_j(M), \mathbb{Z}) = 0$ for $j \geq k+1$. Thus $H_j(M) = 0$ for all j. By Whitehead's theorem, M is contractible. □

The effect of surgery on the middle dimension is more delicate. Let $M' = \chi(M, \varphi)$ with $\varphi : S^k \times \mathbb{D}^{k+1} \to M$ an embedding. We get a corresponding embedding $\varphi' : \mathbb{D}^{k+1} \times S^k \to M'$. Let $\lambda \in H_kM$ corresponding to the homotopy class of $\varphi|_{S^k \times \{0\}}$ and $\lambda' \in H_kM'$ corresponding to $\varphi'|_{\{0\} \times S^k}$. Define maps

$$\cdot\lambda : H_{k+1}M \to \mathbb{Z} \text{ and } \cdot\lambda' : H_{k+1}M' \to \mathbb{Z}$$

by taking the intersection number with λ or λ'.

Proposition 6.7. *Let M be an odd dimensional manifold. There is an isomorphism $H_kM/\lambda(\mathbb{Z}) \cong H_kM'/\lambda'(\mathbb{Z})$.*

Proof. Say $\dim(M) = 2k+1$. Let M_0 be the space obtained from M by removing the interior of $\mathrm{Im}\,\varphi$. We will show that there is a commutative diagram

$$\begin{tikzcd}
 & & H_{k+1}M' \arrow[d, "\cdot\lambda'"] & & \\
 & & \mathbb{Z} \arrow[d, "\epsilon"] \arrow[dr, "\lambda"] & & \\
H_{k+1}M \arrow[r, "\cdot\lambda"] & \mathbb{Z} \arrow[r, "\epsilon'"] \arrow[dr, "\lambda'"'] & H_kM_0 \arrow[r, "i"] \arrow[d, "i'"] & H_kM \arrow[r] & 0 \\
 & & H_kM' \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

with exact column and row. It will follow that $H_kM/\lambda(\mathbb{Z}) \cong H_kM'/\lambda'(\mathbb{Z})$.

For this, take the row to be the long exact sequence for the pair (M, M_0) and the column to be the long exact sequence of the pair (M', M_0). By excision, both $H_j(M, M_0)$ and $H_j(M', M_0)$ are isomorphic to

$$H_j(S^k \times \mathbb{D}^{k+1}, S^k \times S^k) = \begin{cases} \mathbb{Z} & j = k+1 \\ 0 & j < k+1 \end{cases}$$

Since a generator of $1 \in H_{k+1}(M, M_0)$ has intersection number ± 1 with λ, we can describe the map $H_{k+1}M \to \mathbb{Z}$ by $\cdot\lambda$. The image of $1 \in H_{k+1}(M, M_0)$ in $H_k M_0$ is the homology class of $\varphi(\{x_0\} \times S^k)$. In M', the homology class of $\varphi(\{x_0\} \times S^k)$ is λ'. Thus $i'\epsilon' = \lambda'$. The analogous statements hold for λ, i, and ϵ. ☐

We have a similar statement in even dimensions,

Lemma 6.8. *If* $\dim(M) = 2k$*, then we have a diagram*

$$\begin{tikzcd}
& \mathbb{Z} \arrow[d] \arrow[dr, "\lambda"] & & & & \\
0 \arrow[r] & H_k(M_0) \arrow[r] \arrow[d] & H_k(M) \arrow[r, "\cdot\lambda"] & \mathbb{Z} \arrow[r] & H_{k-1}(M_0) \arrow[r] & 0 \\
& H_k(M') \arrow[d] & & & & \\
& 0 & & & &
\end{tikzcd}$$

with exact rows and columns.

Corollary 6.9. *Let* $\dim(M) = 2k+1$*. Any primitive element of* $H_k M$ *can be killed by spherical modification.*

If $\dim(M) = 2k+1$, an element $\lambda \in H_k M$ is primitive if $\mu \cdot \lambda = 1$ for some $\mu \in H_{k+1}M$.

Proof. With notation as above, $H_{k+1}M \xrightarrow{\cdot\lambda} \mathbb{Z} \xrightarrow{\epsilon'} H_k M_0$ is exact. Since λ is primitive, $\cdot\lambda$ is surjective. Thus $\ker(\epsilon') = \mathrm{Im}(\cdot\lambda) = \mathbb{Z}$ and $\epsilon' = 0$. Since $\lambda' = i' \circ \epsilon'$, we also have $\lambda' = 0$. Thus

$$H_k M' = H_k M'/\lambda'(\mathbb{Z}) = H_k M/\lambda(\mathbb{Z})$$

☐

The even dimensional analogue is

Corollary 6.10. *Let* $\dim(M) = 2k$ *and* $\lambda \in H_k M$*. If there exists* $\mu \in H_k M$ *with* $\mu \cdot \lambda = 1$*, then* $H_{k-1}M_0 = 0$*.*

7. The groups bP_{2k+1}

The point of this section is to show that for $n \geq 5$ odd, $bP_n = 0$. Let $k \geq 2$. The group bP_{2k+1} is the image of the map $b : P_{2k+1} \to \Theta_{2k}$. We will show that $P_{2k+1} = 0$. This will be done by showing that a $(2k+1)$-manifold M with boundary a homotopy sphere can be transformed into a contractible manifold by a finite number of framed surgeries. By Lemma 6.2, we can assume M is $(k-1)$connected. By Lemma 6.6, it suffices to show that $H_k M = 0$. We want to be able to kill every $\lambda \in H_k M$ by surgery. By Lemma 6.3, there exists an embedding $\varphi : S^k \times \mathbb{D}^{k+1} \to M$ corresponding to λ. We can therefore always do surgery on λ, the question is whether the resulting homology group $H_k(\chi(M, \varphi))$ will be smaller.

Proposition 7.1. *Let $M \in P_{2k+1}$ be $(k-1)$-connected. Then there exists $M' \in P_{2k+1}$ with $[M] = [M']$, M' still $(k-1)$-connected, and $H_k M'$ isomorphic to the torsion subgroup of $H_k M$.*

Proof. Say $H_k M \cong \mathbb{Z}^{\oplus n} \oplus T$ where T is the torsion subgroup. We will show that each copy of $\mathbb{Z}$ is generated by a primitive element, and hence can be killed by surgery. Let λ generate one of the copies of $\mathbb{Z}$. Then by Poincaré duality, there exists $\mu_1 \in H_{k+1}(M, \partial M)$ so that $\mu_1 \cdot \lambda = 1$. To show λ is primitive, we need to lift μ_1 to an element in $H_{k+1}(M)$. Now

$$H_{k+1}M \to H_{k+1}(M, \partial M) \to H_k(\partial M)$$

is exact. Since ∂M is a homology $2k$-sphere, $H_k(\partial M) = 0$. Thus λ is primitive. After killing finitely many primitive elements, we have $H_k M' = T$. $\square$

The rest of the computation of P_{2k+1} differs for k even and odd.

7.1. k **even.** Let k even and $M \in P_{2k+1}$ be $(k-1)$-connected. Let $\varphi : S^k \times \mathbb{D}^{k+1} \to M$ any embedding.

Lemma 7.2. *The modification $\chi(\varphi)$ changes the kth Betti number of M.*

The proof requires an additional lemma.

Definition. Let F a field and W an orientable homology manifold of dimension $2r$. The semi-characteristic of W is

$$e^*(\partial W; F) := \sum_{i=1}^{r-1} \mathrm{rank} H_i(\partial W; F) \mod 2$$

Let $e(W)$ denote the Euler characteristic of W.

Lemma 7.3. *The rank of the bilinear pairing*

$$H_r(W; F) \otimes H_r(W; F) \to F$$

is congruent to $e^*(\partial W; F) + e(W) \mod 2$.

Proof. Everything is with F coefficients. Consider the long exact sequence of the pair $(W, \partial W)$,

$$\cdots \to H_r W \xrightarrow{h} H_r(W, \partial W) \to H_{r-1}(\partial W) \to \cdots \to H_0(W, \partial W) \to 0$$

Cut of this long exact sequence at $\mathrm{Im}(h)$,

$$0 \to \mathrm{Im}(h) \to H_r(W, \partial W) \to H_{r-1}(\partial W) \to \cdots$$

Then mod 2,

$$\mathrm{rank}(\mathrm{Im}\, h) = \mathrm{rank} H_r(W, \partial W) + \sum_{i=0}^{r-1} \mathrm{rank} H_i W + \mathrm{rank} H_i(W, \partial W) + \mathrm{rank} H_i(\partial W)$$

Since $\mathrm{rank} H_i(W, \partial W) = \mathrm{rank} H_{2r-i} W$, this becomes

$$\mathrm{rank}(\mathrm{Im}\, h) = \sum_{i=0}^{2r} \mathrm{rank} H_i W + \sum_{i=0}^{r-1} \mathrm{rank} H_i(\partial W) = e^*(\partial W) + e(W)$$

Since the rank of h is the rank of the intersection pairing, we are done. $\square$

Proof of Lemma 7.2. Attaching a disk along the boundary of M, we obtain a new manifold M_* so that $\text{rank}H_k(M_*) = \text{rank}H_k(M)$ and $\text{rank}H_k(\chi(M_*, \phi)) = \text{rank}H_k(\chi(M, \phi))$. Thus we can assume M has no boundary. Let $M' = \chi(M, \phi)$ and W be the trace of $\chi(\phi)$. Then W is a $(2k+2)$-dimensional manifold with $\partial W = M \sqcup M'$ that has the homotopy type of M with a $(k+1)$-cell attached. Hence

$$e(W) = e(M) + (-1)^{k+1} = (-1)^{k+1}$$

since $e(M) = 0$ as M is odd dimensional. The intersection pairing

$$H_{k+1}(W; \mathbb{Q}) \otimes H_{k+1}(W; \mathbb{Q}) \to \mathbb{Q}$$

is skew-symmetric since $k+1$ is odd. Hence, the rank of this intersection pairing must be even (cf. [14, Thm. 8.1]). Now by Lemma 7.3,

$$\begin{aligned} 0 &= e^*(\partial W; \mathbb{Q}) + 1 \mod 2 \\ 1 &= e^*(M) + e^*(M') \mod 2 \\ 1 &= \sum_{i=1}^{k} \text{rank}H_i(M; \mathbb{Q}) + \sum_{i=1}^{k} \text{rank}H_i(M'; \mathbb{Q}) \mod 2 \end{aligned}$$

By Lemma 6.2, $H_i(M) = H_i(M')$ for $0 \le i \le k-1$. Hence

$$\text{rank}H_k(M; \mathbb{Q}) \neq \text{rank}H_k(M'; \mathbb{Q})$$

□

Note that the only place we used k even was to show that the intersection pairing had even rank.

Theorem 7.4. *For k even, $bP_{2k+1} = 0$.*

Proof. Let $M \in P_{2k+1}$. It suffices to show that H_kM can be killed. By Proposition 7.1, we can assume H_kM is all torsion. Let $\lambda \in H_kM$. By Lemma 6.3, there exists a corresponding embedding $\varphi : S^k \times \mathbb{D}^{k+1} \to M$. Say $M' = \chi(M, \varphi)$ and λ replaced by λ'. Then by Proposition 6.7,

$$H_k(M)/\lambda(\mathbb{Z}) \cong H_kM'/\lambda'(\mathbb{Z})$$

Since H_kM is all torsion, $\lambda(\mathbb{Z})$ is finite. By Lemma 7.2, H_kM' has smaller rank than H_kM. Thus $\lambda(\mathbb{Z})$ must be infinite. Hence

$$0 \to \mathbb{Z} \xrightarrow{\lambda} H_kM' \to H_kM'/\lambda(\mathbb{Z}) \to 0$$

is exact. Let $x \in H_kM'$ a torsion element. Then the image of x in $H_kM'/\lambda'(\mathbb{Z})$ cannot be zero since $x \notin \text{Im}\,\lambda'$. Thus the torsion subgroup $T' \subset H_kM'$ injects into $H_kM'/\lambda'(\mathbb{Z}) \cong H_kM/\lambda(\mathbb{Z})$ and must be smaller than H_kM. Now using Proposition 7.1, we can perform surgery on M' to obtain M'' with $H_kM'' = T'$ strictly smaller than H_kM. Repeating this finitely many times, we can kill all of H_kM. □

7.2. k **odd.** Let k odd. Assume $M \in P_{2k+1}$ is $(k-1)$-connected and H_kM is finite. As in the even case, we want to perform surgery on M to shrink H_kM. Let $\chi(\varphi)$ be a framed spherical modification that replaces $\lambda \in H_kM$ of order l with $\lambda' \in H_k\chi(M, \varphi)$ of order l'.

Proposition 7.5. *If $l'/l \neq 0 \mod 1$, then there exists $\alpha \in \pi_kSO(k+1)$ so that $\chi(\varphi_\alpha)$ can still be framed and $H_k\chi(M, \varphi_\alpha)$ is smaller than H_kM.*

The proof will break up into two parts:

Part 1: Studying the restrictions on the homology shrinking.

Part 2: Studying the restrictions on $\chi(\varphi_\alpha)$ still being framed.

There will be a lot of notation in this proof which we collect here:

$M'_\alpha = \chi(M, \varphi_\alpha)$,
λ'_α the element $\chi(\varphi_\alpha)$ replaces λ with,
$M_0 = M \setminus (\mathrm{int}\varphi_\alpha(S^k \times \mathbb{D}^{k+1}))$,
$x_0 \in S^k$ a fixed base point,
$j : SO(k+1) \to S^k$ by $j(\rho) = \rho(x_0)$,
$\epsilon' \in H_k M_0$ the homology class corresponding to $\varphi_\alpha(\{x_0\} \times S^k)$,
$\epsilon \in H_k M_0$ the homology class corresponding to $\varphi(S^k \times \{x_0\})$,
$\epsilon_\alpha \in H_k M_0$ the homology class corresponding to $\varphi_\alpha(S^k \times \{x_0\})$.
where the dependence (or lack thereof) on α is indicated in the notation.

Part 1. By Proposition 6.7, we have $H_k M/\lambda(\mathbb{Z}) \cong H_k M'_\alpha/\lambda'_\alpha(\mathbb{Z})$ so that the homology shrinks if and only if the orders of λ'_α and λ satisfy $0 < |\lambda'_\alpha| < |\lambda| = l$. We therefore need to study the order of λ'_α in terms of the orders l, l' of λ and λ'. Looking at the diagram in the proof of Proposition 6.7 for $\chi(\varphi)$, we have $l\epsilon = -l'\epsilon'$. Moreover, ϵ and ϵ' satisfy no other relations. Similarly, looking at the diagram in Proposition 6.7 for $\chi(\varphi_\alpha)$, we see that $|\lambda'_\alpha|$ is the unique integer k so that $l\epsilon_\alpha = -k\epsilon'$.

Now ϵ_α, ϵ, and ϵ' are the homology classes of the following sets

$$\varphi_\alpha(S^k \times \{x_0\}) = \{\varphi(u, j(\alpha(u)))\}_{u \in S^k}$$
$$\varphi(S^k \times \{x_0\}) = \{\varphi(u, x_0)\}_{u \in S^k}$$
$$\varphi_\alpha(\{x_0\} \times S^k) = \{\varphi(x_0, u)\}_{u \in S^k}$$

From these descriptions, we get the equation $\epsilon_\alpha = \epsilon + j_*(\alpha)\epsilon'$ where $j_* : \pi_k SO(k+1) \to \pi_k S^k \cong \mathbb{Z}$ induced by j. We have a string of equalities:

$$\begin{aligned} \epsilon_\alpha &= \epsilon + j_*(\alpha)\epsilon' \\ l\epsilon_\alpha &= l\epsilon + j_*(\alpha)l\epsilon' \\ l\epsilon_\alpha &- l'\epsilon' + j_*(\alpha)l\epsilon' \\ l\epsilon_\alpha &- (l' - lj_*(\alpha))\epsilon' \end{aligned}$$

so that $|\lambda'_\alpha| = l' - lj_*(\alpha)$. Thus $\chi(\varphi_\alpha)$ shrinks the kth homology group if and only if

$$0 < |l' - lj_*(\alpha)| < l$$

The only dependency here on α is $j_*(\alpha)$. We want to know what values of $j_*(\alpha)$ we can get while keeping $\chi(\varphi_\alpha)$ framed.

Part 2. With notation as in Proposition 6.4, the obstruction to $\chi(\varphi_\alpha)$ being framed is

$$\gamma(\varphi_\alpha) = \gamma(\varphi) + s_*(\alpha) = s_*(\alpha)$$

Thus $\chi(\varphi_\alpha)$ can still be framed as long as $\alpha \in \ker(s_*)$. The map s is the inclusion of a fiber

$$SO(k+1) \xrightarrow{s} SO(k+2) \xrightarrow{j'} S^{k+1}$$

with $j'(\rho) = \rho(x'_0)$ where x'_0 is the base point of S^{k+1} given by $(x_0, 0)$. We therefore have an exact sequence

$$\pi_{k+1} S^{k+1} \xrightarrow{\partial} \pi_k SO(k+1) \xrightarrow{s_*} \pi_k SO(k+2)$$

So $\chi(\varphi_\alpha)$ can be framed if and only if $\alpha \in \mathrm{Im}\,\partial$. Say $\alpha = \partial\beta$. We want to study $j_*(\partial\beta)$. Consider the composition

$$\mathbb{Z} = \pi_{k+1} S^{k+1} \xrightarrow{\partial} \pi_k SO(k+1) \xrightarrow{j_*} \pi_k S^k = \mathbb{Z}$$

Elements of $\pi_k SO(k+1)$ correspond to rank $k+1$ bundles on S^{k+1}. Under ∂, the identity map $S^{k+1} \to S^{k+1}$ corresponds to the bundle TS^{k+1}. The map j_* takes a vector bundle ξ to the obstruction $\mathfrak{o}(\xi)$ to having a section. Indeed, ξ has an section if and only if there is a lift

$$\begin{tikzcd} & SO(k) \arrow[d] \\ S^k \arrow[ur, dashed] \arrow[r, "\xi"'] & SO(k+1) \end{tikzcd}$$

since $SO(k) \to SO(k+1) \xrightarrow{j} S^k$ is a fiber bundle, such a lift of ξ exists if and only if the composite $j\xi$ is nullhomotopic.

Now $j_*(\partial(1)) = \mathfrak{o}(TS^{k+1})$ is the obstruction to finding a section of the tangent bundle TS^{k+1}. By an argument similar to the proof given in Lemma 12.13, the obstruction in cohomology is $w_{k+1}(S^{k+1})$. Thus $\mathfrak{o}(TS^{k+1}) = \langle w_{k+1}(TS^{k+1}), [S^{k+1}]\rangle$ is the Euler characteristic $e(TS^{k+1})$. When k is odd, $e(S^{k+1}) = 2$. Thus $j_* \circ \partial$ is multiplication by ± 2.

Hence the surgery $\chi(\varphi_\alpha)$ can be framed as long as the element

$$j_*(\alpha) = j_*(\partial\beta) \in \mathbb{Z}$$

is even. Thus for any even $n \in \mathbb{Z}$, there exists φ_α so that $\chi(\varphi_\alpha)$ is framed and $j_*(\alpha) = n$.

Conclusion. We want α so that $\chi(\varphi_\alpha)$ is framed and $0 < |l' - lj_*(\alpha)| < l$. Since we are assuming that l' does not divide l, we have $nl < l' < (n+1)l$ for some $n \in \mathbb{Z}$. Either n or $(n+1)$ is even, say n is. Choose α so that $\chi(\varphi_\alpha)$ is framed and $j_*(\alpha) = n$. Then

$$|l' - lj_*(\alpha)| = |l' - ln| < l$$

so that by Part 1, $H_k M'_\alpha$ is strictly smaller than $H_k M$. $\square$

Our next step is to give a better description of $l'/l \mod 1$. Consider the intersection pairing

$$\cdot : H_{p+1}(M; \mathbb{Q}/\mathbb{Z}) \otimes H_q(M;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

induced by multiplication $\mathbb{Q}/\mathbb{Z} \otimes \mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$. Also note that the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

induces a long exact sequence

$$\cdots \to H_{p+1}(M;\mathbb{Q}/\mathbb{Z}) \to H_p(M;\mathbb{Z}) \to H_p(M;\mathbb{Q}) \to \cdots$$

Definition. Let $\lambda \in H_p M$ and $\mu \in H_q M$ of finite order with $\dim M = p+q+1$. Since λ is torsion, it lifts to an element $\hat{\lambda} \in H_{p+1}(M;\mathbb{Q}/\mathbb{Z})$. Define the linking number of λ and μ to be

$$L(\lambda,\mu) = \hat{\lambda}\cdot\mu \in \mathbb{Q}/\mathbb{Z}$$

We will need the following properties of the linking number. For a proof see [25, §77].

Lemma 7.6. *The linking number is well-defined and satisfies*

$$L(\mu,\lambda) + (-1)^{pq} L(\lambda,\mu) = 0$$

In particular, if $p = q = k$ is odd, then $L(\mu,\lambda) = L(\lambda,\mu)$.

Lemma 7.7. *The quotient l'/l is the self-linking number, $l'/l = L(\lambda,\lambda) \mod 1$.*

Proof. Let $c_1 \in H_{p+1}(M; \mathbb{Q}/\mathbb{Z})$ denote the homology class of $\phi(\{x_0\} \times \mathbb{D}^{k+1})$. Since $\varphi(\{x_0\} \times S^k)$ bounds $\varphi(\{x_0\} \times \mathbb{D}^{k+1})$, we have $\partial c_1 = \epsilon'$. Since $l\epsilon = -l'\epsilon'$, we have $\partial(-l'/lc_1) = \epsilon$. Thus

$$L(\lambda, \lambda) = (-l'/l)c_1 \cdot \lambda = l'/l \mod 1$$

since $c_1 \cdot \lambda = \pm 1$. □

Thus when $L(\lambda, \lambda) \neq 0$, we can remove λ by a (framed) surgery and obtain a manifold with smaller kth homology group. We are thus left considering the case $L(\lambda, \lambda) = 0$ for all $\lambda \in H_k(M)$.

Lemma 7.8. *If $L(\lambda, \lambda) = 0$ for all $\lambda \in H_k(M)$, then $H_k M$ must be a direct sum of cyclic groups of order 2. In particular, $H_k(M; \mathbb{Z}) = H_k(M; \mathbb{Z}/2)$.*

Proof. Fix $\xi \in H_k M$. Let $\eta \in H_k M$ any class. Using Lemma 7.6 and our assumption that all self-linking numbers are zero, we have

$$0 = L(\xi + \eta, \xi + \eta) = L(\xi, \xi) + L(\eta, \eta) + 2L(\xi, \eta) = 2L(\xi, \eta)$$

Since the pairing L is nondegenerate and this is true for all η, we must have $2\xi = 0$. □

Let $\chi(\varphi_\alpha)$ be a framed surgery. Then the trace W of $\chi(\varphi_\alpha)$ is parallelizable. Hence the $(k+1)$th Wu class $v_{k+1} = 0$. Hence

$$Sq^{k+1} : H^{k+1}(W, \partial W; \mathbb{Z}/2) \to H^{2k+1}(W, \partial W; \mathbb{Z}/2)$$

by $\mu \mapsto v_{k+1} \cup \mu$ is zero. Hence $\xi \cdot \xi = 0$ for all $\xi \in H_{k+1}(W; \mathbb{Z}/2)$. We are now in position to apply the following analogue of Lemma 7.2.

Lemma 7.9. *Let W be the trace of the surgery $\chi(\varphi)$ on M. Suppose that every mod 2 homology class $\xi \in H_{k+1}(W; \mathbb{Z}/2)$ has self-intersection number $\xi \cdot \xi = 0$. Then $\chi(\varphi)$ necessarily changes the rank of $H_k(M; \mathbb{Z}/2)$.*

Proof. The assumption implies that the intersection pairing

$$H_{k+1}(W; \mathbb{Z}/2) \otimes H_{k+1}(W; \mathbb{Z}/2) \to \mathbb{Z}/2$$

has even rank. Now the proof given for Lemma 7.2 implies the result. □

Theorem 7.10. *For k odd, $bP_{2k+1} = 0$.*

Proof. It suffices to show that for $(k-1)$-connected $M \in P_{2k+1}$ with $H_k M$ finite, that we can kill all of $H_k M$ by surgery. By Proposition 7.5 and Lemma 7.7, we can assume $L(\lambda, \lambda) = 0$ for all $\lambda \in H_k M$. By Lemma 7.8, we can assume $H_k M$ is a direct sum of copies of $\mathbb{Z}/2$. Finally, by Lemma 7.9, we can kill all of $H_k M$ by a finite number of surgeries.

□

8. The groups bP_{2k}

The goal of this section is to compute the order of $bP_{2k} \subset \Theta_{2k-1}$. In particular, we want to show that it is a finite subgroup. The methods used here will not work in dimension 4, so throughout we assume $k \geq 3$.

The main tool used here will be the intersection form. We will denote the intersection form on M by Q_M or Q when the manifold in question is clear. Our proof will break into two cases, k even or k odd. In the even case, we will use the signature of Q_M. In the odd case, we will use the Kervaire invariant. Before breaking into two cases, we will prove the main technical lemma that will be used in both cases.

Let $M \in P_{2k}$ with $k \geq 3$. By Lemma 6.2, we can assume M is $(k-1)$-connected. By Poincaré duality, it follows that $H_k M$ is free abelian. To show that M is trivial, we want to kill $H_k M$

using surgery (cf. Lemma 6.6). Unlike in the case of bP_{2k+1}, elements in H_kM might not have corresponding embeddings $\varphi : S^k \times \mathbb{D}^k \to M$. The problem here is that Lemma 6.3 requires k to be *strictly* less than half the dimension of M. The restriction on k was used to show that the normal bundle of a representative embedded sphere $\lambda : S^k \to M$ was trivial. The tubular neighborhood theorem then gave us an embedding φ. Our main challenge is to determine when the normal bundles of embedded spheres are trivial. Once we perform surgery on a given element of H_kM, we want embedded spheres in the new manifold $\chi(M, \varphi)$ to have trivial normal bundles as well. The following lemma describes when this occurs.

Lemma 8.1. *Let M be a $(k-1)$-connected $2k$-manifold with $k \geq 3$. Suppose H_kM is free abelian with basis $\{\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_r\}$ with $\lambda_i \cdot \lambda_j = 0$ and $\lambda_i \cdot \mu_j = \delta_{ij}$. If every embedded sphere in M which represents a λ_i has trivial normal bundle, then H_kM can be killed by a finite sequence of surgeries.*

Such a basis is called weakly symplectic. If we additionally require $\mu_i\mu_j = 0$, the basis is called symplectic.

Proof. Let $\varphi_0 : S^k \to M$ represent λ_r. Since the normal bundle of f_0 is trivial, the tubular neighborhood theorem gives us an embedding $\varphi : S^k \times \mathbb{D}^k \to M$. We will show that performing surgery on M using φ results in a manifold $M' = \chi(M, \varphi)$ satisfying the hypothesis of 8.1 but with smaller basis $\{\lambda'_1, \ldots, \lambda'_{r-1}, \mu'_1, \ldots, \mu'_{r-1}\}$ for H_kM'. By a finite number of surgeries, we will be able to kill all of H_kM.

Let $M_0 = M \setminus (\mathrm{Int}\varphi)$. By Lemma 6.8 we have a diagram with exact rows and columns,

$$\begin{tikzcd}
 & \mathbb{Z} \arrow[d] \arrow[dr, "\lambda"] & & & & \\
0 \arrow[r] & H_k(M_0) \arrow[r] \arrow[d] & H_k(M) \arrow[r, "\cdot\lambda"] & \mathbb{Z} \arrow[r] & H_{k-1}(M_0) \arrow[r] & 0 \\
 & H_k(M') \arrow[d] & & & & \\
 & 0 & & & &
\end{tikzcd}$$

By Corollary 6.10, we have $H_{k-1}M_0 = 0$. Since $H_iM_0 = H_iM$ for $i < k-1$, it follows that M_0 is $(k-1)$-connected. Since $H_jM_0 = H_jM'$ for $j < k$, the new manifold M' is also $(k-1)$-connected. Looking at the exact row, the group H_kM_0 is isomorphic to the subgroup of H_kM generated by $\{\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_{r-1}\}$. Under this identification, H_kM' is isomorphic to the quotient of H_kM_0 by the subgroup generated by λ_r. Let λ'_i, μ_i represent λ_i, μ_i in H_kM' for $i = 1, \ldots, r-1$. Since the λ'_i, μ'_i are coming from homology elements in $H_kM_0 \hookrightarrow H_kM$, it follows that $\{\lambda'_i, \mu'_i\}$ is weakly symplectic. Let $f : S^k \to M'$ an embedded sphere representing some λ'_i. Push $\mathrm{Im}\, f$ into M_0. The deformed $\mathrm{Im}\, f$ represents a homology class in $\mathrm{span}\{\lambda_1, \ldots, \lambda_r\} \subset H_kM$ and hence has trivial normal bundle. $\square$

The invariants for k even and odd will appear at different steps of applying Lemma 8.1.

- For k even, such a basis will exist if the signature of the manifold vanishes. The triviality of the normal bundles will be automatic in this case.
- For k odd, symplectic bases always exist. The invariant comes up in determining when the normal bundles are trivial.

8.1. k **even.** Throughout we will assume $k = 2m > 1$ is even. We will show that $\frac{1}{8}\mathrm{sign} : P_{4m} \to \mathbb{Z}$ defines an isomorphism and then compute $\mathbb{Z}/\ker(b) = bP_{4m}$.

Lemma 8.2. *The signature is a cobordism invariant* $\mathrm{sign} : P_{4k} \to \mathbb{Z}$.

Proof. Since $\mathrm{sign}(A \sqcup B) = \mathrm{sign}(A) + \mathrm{sign}(B)$, it suffices to show that the signature of a boundary is zero. Let $M = \partial W$ be an oriented nullbordant $2k$-manifold with $n = 2k$. We will produce a basis $\{a_i, b_i\}$ for $H^k M$ so that $a_i b_j = \delta_{ij}$ and $a_i a_j = b_i b_j = 0$. The matrix for Q_M will then look like 2×2 matrices $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ on the diagonal and zeros elsewhere. The signature of such a matrix is zero.

Let $i : M \to W$ the inclusion. Then we have a commutative diagram with exact rows

$$\begin{tikzcd} \cdots \arrow[r] & H^r W \arrow[r, "i^*"] \arrow[d, "\cong"] & H^r M \arrow[r, "\delta"] \arrow[d, "\cong"] & H^{r+1}(W,M) \arrow[r] \arrow[d, "\cong"] & H^{r+1} W \arrow[r] \arrow[d, "\cong"] & \cdots \\ \cdots \arrow[r] & H_{n+1-r}(W,M) \arrow[r, "\partial"] & H_{n-r} M \arrow[r, "i_*"] & H_{n-r} W \arrow[r] & H_{n-r}(W,M) \arrow[r] & \cdots \end{tikzcd}$$

where everything is with $\mathbb{R}$ coefficients. Let $A^r = \mathrm{Im}(i^* : H^r W \to H^r M)$ and $K_{n-r} = \ker(i_* : H_{n-r} M \to H_{n-r} W)$. We have a string of equalities

$$A^r = \mathrm{Im}\, i^* = \ker \delta = \ker i_* = K_{n-r}$$

For $a \in A^r$ and $b \in A^{n-r}$ we have

$$\langle a \cup b, [M] \rangle = \langle i^*(a \cup b), \partial[W] \rangle = \langle \delta i^*(a \cup b), [W] \rangle = \langle 0, [W] \rangle = 0$$

Since we are working over a field, $H^i M \cong H_i M$ and i^* is dual to i_*, i.e., we have a commutative diagram

$$\begin{tikzcd} H_{n-p} W \arrow[d, "\cong"] & H_{n-p} M \arrow[l, "i^*"'] \arrow[d, "\cong"] \\ H^{n-p} W \arrow[r, "i^*"] & H^{n-p} M \end{tikzcd}$$

Thus $H_{n-p}M/K_{n-p} \cong (A^{n-})^\vee$. Hence the annihilator of A^{n-p} is exactly $K_{n-p} \cong A^p$. Putting this together, we have $H^k M \cong A \oplus B$ with A and B duals. Take dual bases $\{a_i, b_i\}$ so that $a_i b_j = \delta_{ij}$ and $a_i a_j = b_i b_j = 0$. Calculating the signature using this basis we see that $\mathrm{sign}(M) = 0$. ☐

We have shown that for M^{2k} a boundary, there exists a symplectic basis $\{a_i, b_i\}$ for $H^k M$ with $a_i \in \ker \delta$ for all i. We will use this later (cf. Lemma 8.12).

Since we only care about things up to cobordism, by Lemma 6.2 we can assume M is $(2k-1)$-connected.

Lemma 8.3. *Let $M \in P_{4m}$ and $f_0 : S^{2m} \to M$ an embedding representing $\beta \in H_{2m}(W)$. Then the normal bundle ν of $f_0(S^{2m})$ is trivial if and only if $\beta \cdot \beta = 0$. Additionally, $\beta \cdot \beta$ is always even so that Q_M is even.*

Proof. Since $f_0(S^{2m})$ is stably trivial and M is parallelizable, ν is stably trivial. We can assume $\nu \oplus \epsilon^1$ is trivial. Thus $[\nu] \in \pi_{2m-1} SO(2m)$ maps to zero under the inclusion induced map

$$s_* : \pi_{2m-1}(SO(2m)) \to \pi_{2m-1}(SO(2m+1))$$

Consider the diagram (cf. *Part 2* of the proof of Proposition 7.5)

$$\begin{tikzcd} \pi_{2m}S^{2m} \arrow[r, "\partial"] \arrow[dr, "\pm 2"'] & \pi_{2m-1}SO(2m) \arrow[r, "s_*"] \arrow[d, "j_*"] & \pi_{2m-1}SO(2m+1) \\ & \pi_{2m-1}S^{2m-1} & \end{tikzcd}$$

where the top row is exact, $\partial r = r[TS^{2m}]$ and $j_*([\nu]) = \chi(\nu)$ is the Euler characteristic of ν. Now $s_*([\nu]) = 0$ so $[\nu] = \partial r$ for some $r \in \pi_{2m}S^{2m} = \mathbb{Z}$. Since TS^{2m} is nontrivial, we have ν trivial if and only if $r = 0$, if and only if $\pm 2r = j_*([\nu]) = \chi(\nu)$ is zero. The Euler characteristic $\chi(\nu)$ is $\pm\beta \cdot \beta$. Indeed, we can deform f_0 along a vector field for ν to a new embedding f_0' that intersects $f_0(S^{2m})$ at only finitely many points. At each point, the multiplicity of the intersection of f_0 and f_0' is equal to the index of the vector field at that point. By the Poincaré-Hopf theorem, the sum of these indices, i.e., $\beta \cdot \beta$, is equal to the Euler characteristic $\chi(\nu)$. Thus ν is trivial if and only if $\beta \cdot \beta = 0$ and $\beta \cdot \beta = 2r$ is always even. □

Note that by Poincaré duality, Q_M is unimodular, i.e., $\det(Q_M) = \pm 1$. Now Lemma 8.1 will imply that $[M] \in P_{4k}$ as long as H_kM has a weakly symplectic basis.

Claim. *Let Q be an even unimodular form on H_kM. If* $\text{sign}(Q) = 0$*, then H_kM has a weakly symplectic basis.*

For a proof see [19, Lem. 9].

When $\text{sign}(M) = 0$ Lemma 8.1 implies that H_kM can be killed by surgery and hence $[M] = 0 \in P_{4k}$. In other words, the signature defines an injective homomorphism $\text{sign} : P_{4k} \to \mathbb{Z}$. Our next step is surjectivity. First, we note that $\text{sign}(P_{4k})$ lands in $8\mathbb{Z}$. Indeed, for any $M \in P_{4k}$, Lemma 8.3 implies that Q_M is even and unimodular. By [26, pg. 52], such quadratic forms have signature divisible by 8. Consider the map $\frac{1}{8}\text{sign} : P_{4k} \to \mathbb{Z}$.

Theorem 8.4. *For any $t \in \mathbb{Z}$, there exists a manifold $M \in P_{4k}$ so that* $\text{sign}(M) = 8t$.

Proof. Since sign is a group map, it suffices to find a manifold $M \in P_{4k}$ with $\text{sign}(M) = 8$. Let E_8 be the following matrix

$$E_8 = \begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & 0 & 1 \\ & & & & 1 & 2 & 1 & 0 \\ & & & & 0 & 1 & 2 & 0 \\ & & & & 1 & 0 & 0 & 2 \end{bmatrix}$$

where the blank spots are zeroes. By [7, Lem V.2.8], $\det(E_8) = 1$ and $\text{sign}(E_8) = 8$.

Choose eight $2m$-spheres $S_i = S_i^{2m}$ and consider their disk bundles $\mathsf{Disk}(TS_i) = \mathsf{Disk}(TS_i)$. We will "plumb" together these disk bundles using E_8. Let a_{ij} denote the ijth entry of E_8. For each $a_{ij} \neq 0$ with $i \neq j$ glue $\mathsf{Disk}(TS_i)$ and $\mathsf{Disk}(TS_j)$ together as follows

- For $s = i, j$ choose disks $D_s = \mathbb{D}_s^{2m} \subset S_s$ in the bases and trivializations

$$\mathsf{Disk}(TS_s)|_{D_s} \cong D_s \times \mathbb{D}^{2m}.$$

- Glue $D_i \times \mathbb{D}^{2m}$ to $\mathbb{D}^{2m} \times D_j$ by switching order $(u, v) \sim (v, u)$.

After smoothing things out, we get a $4m$-manifold M. We wanted $M \in P_{4k}$ with signature 8. We therefore need to show

1) $\mathrm{sign}(M) = 8$,
2) M is parallelizable, and
3) ∂M is a homotopy sphere.

(1) The inclusions $i(S_i)$ into M define a basis of $H_{2k}M$. By construction, the intersection number $i(S_i) \cdot i(S_j) = a_{ij}$ for $i \neq j$. The self-intersection numbers $i(S_i) \cdot i(S_i)$ are all equal to 2 since they are coming from one copy of TS^{2m} (cf. the proof of Lemma 8.3). Thus the intersection form Q_M has matrix E_8 and $\mathrm{sign}(M) = 8$.

To show that $M \in P_{4k}$ it is helpful to note that M has the homotopy type of a wedge of $2k$-spheres. To see this, note that each disk bundle $\mathsf{Disk}(TS_s)$ deformation retracts onto S_s. Inside M, $\mathsf{Disk}(TS_s) \cap \mathsf{Disk}(TS_s)$ is either empty or a single disk. Deforming all of the disk bundles in M down to spheres and their intersection to points, we see that M deformation retracts onto a union $\bigcup S_s$ with $S_s \cap S_r$ ether empty or 1 point. The claim follows.

(2) Since M has the homotopy type of a wedge, it suffices to show that TM is trivial on each piece of the wedge. Let $\pi : TS_i \to S_i$ be the projection. Since spheres are stably parallelizable,

$$TM|_{\mathsf{Disk}(TS_i)} \oplus \epsilon^1 = \pi^* TS_i \oplus \pi^*(\mathsf{Disk}(TS_i)) \oplus \epsilon^1 = \pi^* TS_i \oplus \epsilon^{2k+1} = \epsilon^{4k+1}$$

It follows that $TM|_{S_i}$ is stably trivial. Hence TM is stably trivial. By Corollary 3.3, M is parallelizable.

(3) It suffices to show that the $(4k-1)$-manifold ∂M is $2k$-connected. Since M is a wedge of $2k$-spheres, we have $H_i M = 0$ for $i \neq 2k$ and $H_{2k}M = \mathbb{Z}^{\oplus 8}$. By the universal coefficient theorem, the groups $H^i M = 0$ for $i \neq 2k$ and are free for $i = 2k$. By Poincaré duality, $H^i(M, \partial M) = H_i(M, \partial M) = 0$ for $i \neq 2k$ and are free for $i = 2k$. Looking at the exact sequences for the pair $(M, \partial M)$, it follows that $H^i \partial M = H_i \partial M = 0$ for $i \neq 0, 2k-1, 2k, 4k-1$. We have a commutative diagram with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & H^{2k-1}\partial M \arrow[r] \arrow[d, "\cong"] & H^{2k}(M,\partial M) \arrow[r, "j^*"] \arrow[d, "\cong"] & H^{2k}M \arrow[r] \arrow[d, "\rho"', "\cong"] & H^{2k}\partial M \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & H_{2k}\partial M \arrow[r] & H_{2k}M \arrow[r, "j_*"] & H_{2k}(M,\partial M) \arrow[r] & H_{2k-1}\partial M \arrow[r] & 0
\end{tikzcd}$$

To show that $H_{2k}\partial M = H_{2k-1}\partial M = 0$, and hence that ∂M is a homotopy sphere, it suffices to show that j_* is an isomorphism. Equivalently, we need to show that

$$\rho^{-1} j_* : H_{2k}M \to H^{2k}M \cong \mathrm{Hom}(H_{2k}M, \mathbb{Z})$$

is an isomorphism. Here the last isomorphism comes from the fact that $H_{2k}M$ is free. But $\rho^{-1}j_*(\alpha)$ is the map $\beta \mapsto \alpha \cdot \beta$ and hence is represented by the matrix Q_M. Since $Q_M = E_8$ is unimodular, the map $\rho^{-1} j_*$ is invertible. Thus ∂M is $(2k-2)$-connected and hence a homotopy sphere. $\square$

Thus we have an isomorphism $\frac{1}{8}\mathrm{sign} : P_{4k} \to \mathbb{Z}$. By Theorem 5.5 we have a long exact sequence

$$\cdots \to \Omega_{4k}^{alm} \xrightarrow{R} P_{4k} \xrightarrow{b} \Theta_n \to \cdots$$

So that $bP_{4k} \cong \mathbb{Z}/\operatorname{Im} R$. To see that bP_{4k} is finite, we just need $\operatorname{Im} R$ to be nontrivial. Composing with the isomorphism $\frac{1}{8}\mathrm{sign}$, we need an almost stably framed $4k$-manifold whose signature is nonzero.

Lemma 8.5. *For every k, there exists $N \in \Omega_{4k}^{alm}$ so that* $\mathrm{sign}(N) \neq 0$.

Proof. By Lemmas 3.10 and 3.11, it suffices to produce $N \in \Omega_{4k}^{alm}$ with corresponding obstruction class $\mathfrak{o} \in \pi_{4k-1}SO$ nonzero. Consider the J-homomorphism $J : \mathbb{Z} = \pi_{4k-1}SO \to \pi_{4k-1}^S$. By a result of Serre [27], π_{4k-1}^S is a finite group. Thus there exists some nonzero $\mathfrak{o} \in \pi_{4k-}SO$ so that $J\mathfrak{o} = 0$. Since $J\mathfrak{o} = 0$, the long exact sequence from Lemma 3.6 implies that $\mathfrak{o} = \partial N$ for some almost stably framed manifold $N \in \Omega_{4k}^{alm}$. By definition of ∂, the element ∂N is the obstruction to extending the almost stable framing of N to a stable framing. □

Thus the group bP_{4k} is finite. To determine the exact order of bP_{4k}, we need to know the order of $\operatorname{Im} R$. What integers can appear as the signature of an almost stably framed $4k$-manifold?

Theorem 8.6. *Let $N \in \Omega_{4k}^{alm}$. Then* $\operatorname{sign}(N)$ *is a multiple of* $a_k s_k (2k-1)! |\operatorname{Im} J|$

Notation here is as in §3.

Proof. By Lemmas 3.11 and 3.10, we have

$$\operatorname{sign}(N) = s_k \langle p_k(N), [N] \rangle = \pm s_k a_k (2k-1)! \langle \mathfrak{o}(N), [N] \rangle$$

Viewing the obstruction $\mathfrak{o}(N)$ as an element of $\pi_{4k-1}SO$, the map $\partial : \Omega_{4k}^{alm} \to \pi_{4k-1}SO$ takes N to $\mathfrak{o}(N)$ (cf. §3.1). By Lemma 3.6 and Lemma 3.7, we have an exact sequence

$$\Omega_{4k}^{alm} \xrightarrow{\partial} \pi_{4k-1}SO \xrightarrow{J} \pi_{4k-1}^S$$

so that the number of possible values of $\mathfrak{o}(N) \in \pi_{4k-1}SO \cong \mathbb{Z}$ is the size of $\operatorname{Im} \partial = \ker J$. Now an element in $\pi_{4k-1}SO$ is in the kernel of J if and only if it is a multiple of $|\operatorname{Im} J|$. □

In [1], Adams computed the size of the image of $J : \pi_{4k-1}SO \to \pi_{4k-1}^S$ to be the denominator of $B_k/4k$, where B_k is the kth Bernoulli number.[3] Thus the subgroup $bP_{4k} \subset \Theta_{4k}$ has order

$$\frac{a_k s_k (2k-1)!}{8} \cdot \text{denominator}(B_k/4k) = \frac{3-(-1)^k}{2} \cdot 2^{2k-2}(2^{2k-1}-1) \cdot \text{numerator}(B_k/4k)$$

8.2. k **odd.** To make use of Lemma 8.1, we need a certain type of basis for $H_k M$.

Lemma 8.7. *For k odd, Q_M has a symplectic basis $\{\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_r\}$ with $\lambda_i \cdot \lambda_j = 0 = \mu_i \cdot \mu_j$ and $\lambda_i \cdot \mu_j = \delta_{ij}$*

Proof. This follows from the graded commutativity of the cup product. □

Corollary 8.8. *The groups bP_2, bP_6 and bP_{14} are zero.*

Proof. For $k = 1$, we have $bP_2 \subset \Theta_2 = 0$.

Let M in P_{2k} for $k = 3, 7$. Let $\lambda_1, \dots, \lambda_r$ be part of a symplectic basis for $H_k M$ and ν_i be the normal bundle of an embedded sphere representing λ_i, $i = 1, \dots, r$. By Lemma 8.1, it suffices to show that the ν_i are trivial. But the obstruction for ν_i being trivial live in $\pi_{k-1}SO(k) = 0$ for $k = 3, 7$. □

For general k, the normal bundles ν may not be trivial. Throughout the rest of this section, we will assume k is odd, $k \neq 1, 3, 7$, and $M \in P_{2k}$. Define a (set) map $\psi : H_k M \to \mathbb{Z}/2$ as follows. For $\alpha \in H_k M$, let $f : S^k \to M$ be an embedding representing α. Set $\psi(\alpha) = 0$ if the normal bundle of f is trivial. The proofs given here involving ψ are all following [15, §4].

[3] Adams computation required the Adams conjecture which was first proven by Quillen, [24].

Note. Let $\beta \in H_k M$ any homology class. Let $S^k \to M$ an embedding representing β with normal bundle ν. Since TM is trivial and TS^k is stably trivial, ν is stably trivial. Thus the class $[\nu] \in \pi_{k-1}SO(k)$ corresponding to ν is in the kernel of the map

$$\pi_{k-1}SO(k) \to \pi_{k-1}SO$$

For $k \neq 1, 3, 7$ odd, the kernel of this map is $\mathbb{Z}/2$ with nontrivial element $[TS^k]$. So either $\psi(\beta) = 0$ and ν is trivial or $\psi(\beta) = 1$ and $\nu \cong TS^k$. We will use this fact a lot.

Lemma 8.9. *The map $\psi : H_k(M) \to \mathbb{Z}/2$ is well-defined.*

Proof. Let $f_0, f_1 : S^k \to M$ be two embedings representing α. Let $h : S^k \times I \to M$ be a homotopy between f_0 and f_1. Define $H : S^k \times I \to M \times I$ by $H(x,t) = (h(t), t)$. By Whitney's theorem [32], H is homotopic to an immersion keeping the boundary fixed. In other words, we have an immersion $H : S^k \times I \to M \times I$ so that $H_0(x) = (f_0(x), 0)$ and $H_1(x) = (f_1(x), 1)$. Since H is an immersion, it has a normal bundle $\nu(H)$. Now $\nu(H) : S^k \times I \to BO$ is a homotopy between $\nu(f_0)$ and $\nu(f_1)$. □

We need the following result. See [16, Thm. 4.8] for a proof.

Theorem 8.10. *An immersion $f : S^k \to M^{2k}$ is regularly homotopic to an embedding if and only if f has self-intersection number zero.*

Corollary 8.11. *For any $\alpha, \beta \in H_k M$,*

$$\psi(\alpha + \beta) \equiv \psi(\alpha) + \psi(\beta) + \alpha \cdot \beta \mod 2$$

Proof. Let $f, g : S^k \to M$ be embeddings representing α and β, respectively. Then $f\#g : S^k \to M$ is an immersion representing $\alpha + \beta$. Even though $f\#g$ is only an immersion, we can still ask whether or not its normal bundle is trivial. Write $\psi(f\#g) = 0$ or 1 depending on the triviality of the normal bundle. By Lemma 8.9, $\psi(f\#g)$ only depends on the homotopy class of $f\#g$. We have equalities

$$\psi(f\#g) = \psi(f) + \psi(h) = \psi(\alpha) + \psi(\beta)$$

By Theorem 8.10, $\psi(f\#g) = \psi(\alpha + \beta)$ if and only if $f\#g$ has self-intersection number zero. But the self-intersection number of $f\#g$ is the mod 2 value of $\alpha \cdot \beta$. The result follows for $\alpha \cdot \beta \equiv 0 \mod 2$.

If $\alpha \cdot \beta \equiv 1 \mod 2$, then $f\#g$ has self-intersection number 1. Let $h : S^k \to M$ be the composition of the diagonal map $\Delta : S^k \to S^k \times S^k$ with the inclusion of some coordinate chart $S^k \times S^k \subset \mathbb{R}^{2k} \to M$. Then h is a nullhomotopic immersion with self-intersection number 1. Moreover $\psi(h) = 1$ since the normal bundle $\mathrm{Norm}(\Delta)$ of the diagonal is the tangent bundle TS^k and $k \neq 1, 3, 7$. Now $\psi(f\#g\#h)$ vanishes so that

$$\psi(\alpha + \beta) = \psi(f\#g\#h) = \psi(f) + \psi(g) + \psi(h) = \psi(\alpha) + \psi(\beta) + 1$$

□

Definition. Define the Kervaire invariant $c(M)$ of $M \in P_{2k}$ to be $\sum_{i=1}^{r} \psi(\lambda_i)\psi(\mu_i) \in \mathbb{Z}/2$ where $\{\lambda_i, \mu_i\}_{i=1}^r$ is a symplectic basis of $H_k M$.

This is called the Arf invariant of the quadratic form ψ.

Lemma 8.12. *The Kervaire invariant $c : P_{4k} \to \mathbb{Z}/2$ is well-defined, i.e., is an invariant of framed cobordism.*

Proof. Let $(M, \bar{u}) \in P_{2k}$ with $\bar{u}$ a trivialization of TM. We can assume M is $(k-1)$-connected. If M is nullbordant, then there exists a framed $(2k+1)$-manifold W defining a cobordism between M and $\mathbb{D}^{2k}$. After possibly performing surgery on W, we can assume W is $(k-1)$-connected (cf. Lem. 6.2). We want to show that $c(M) = 0$. Let $i : M \to W$ be the inclusion and $\alpha \in H_k M$ represented by $f : S^k \to M$. If $i_*\alpha = 0 \in H_k W$, then $i \circ f$ is nullhomotopic since $H_k V = \pi_k V$ by Hurewicz's theorem. Then there exists an extension $g : \mathbb{D}^{k+1} \to W$ of $i \circ f$. It follows that the normal bundle of f in M is trivial, i.e., that $\psi(\alpha) = 0$.

To show that $c(M) = 0$, it therefore suffices to show that there exists a symplectic basis $\{\lambda_i, \mu_i\}_{i=1}^r$ for $H_k(M)$ so that each $\lambda_i \in \ker i_*$. Dually, it suffices to find a basis $\{u_i, v_i\}_{i=1}^r$ so that

$$u_i \in \ker(\delta : H^k M \to H^{k+1}(W, M))$$

The proof of Lemma 8.2 shows that such a basis exists. □

Lemma 8.13. *The map $c : P_{2k} \to \mathbb{Z}/2$ is injective.*

Proof. Say $c(M) = 0$. We will form a new symplectic basis $\{\lambda_i', \mu_i'\}$ for $H_k M$ so that each λ_i' has trivial normal bundle. Then by Lemma 8.1, $[M] = 0$.

If $\psi(\lambda_i)\psi(\mu_i) = 0$, set

$$(\lambda_i', \mu_i') = \begin{cases} (\lambda_i, \mu_i) & \psi(\lambda_i) = 0 \\ (\mu_i, \lambda_i) & \psi(\mu_i) = 0 \end{cases}$$

Since $c(M) = 0$, the values of i so that $\psi(\lambda_i)\psi(\mu_i) \neq 0$ come in pairs. Say $\psi(\lambda_j)\psi(\mu_j) \neq 0$ as well. Set $(\lambda_i', \mu_i') = (\lambda_i + \lambda_j, \mu_i)$ and $(\lambda_j', \mu_j') = (\mu_j - \mu_i, \lambda_i)$. Then $\{\lambda_i', \mu_i'\}$ is symplectic and $\psi(\lambda_i') = 0$ for all i. □

Thus $b : P_{2k} \to \Theta_{2k-1}$ has image at most a subgroup of order 2, i.e., bP_{2k} is finite for k odd.

Theorem 8.14 (Kervaire)**.** *The map $c : P_{2k} \to \mathbb{Z}/2$ is surjective.*

Proof. Plumb together two disk bundle $\mathsf{Disk}(TS^k)$ using the matrix

$$Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The disk bundles are glued with negative orientation when $a_{ij} = -1$ and positive orientation when $a_{ij} = 1$. Let W be a smoothed version of the resulting manifold. Just as in the proof of Theorem 8.4, W is parallelizable and $\det(Q) = 1$ together imply that ∂W is a homotopy sphere. We want to show $c(W) = 1$. Let $x_i : S^k \to W$, $i = 1, 2$ be the zero sections of the two disk bundles that form W. Then $\{x_1, x_2\} \subset H_k(W)$ form a symplectic basis. The normal bundle of x_i in W is $\mathsf{Disk}(TS^k)$. Since TS^k is nontrivial, $\psi(x_i) = 1$ for $i = 1, 2$. Thus $c(W) = \psi(x_1)\psi(x_2) = 1$. □

From the exact sequence of Theorem 5.5, we have $bP_{2k} = (\mathbb{Z}/2)/\operatorname{Im}(R)$ where $R : \Omega_{2k}^{alm} \to P_{2k}$. Thus $bP_{2k} = \mathbb{Z}/2$ unless there exists $M \in \Omega_{2k}^{alm}$ with Kervaire invariant $c(M) = 1$, in which case $bP_{2k} = 0$.

Theorem 8.15 (Hill, Hopkins, Ravenel [11])**.** *The groups bP_{4m-2} can be described as follows*

$$bP_{4m-2} = \begin{cases} 0 & m = 1, 2, 4, 8, 16 \\ \textit{either } 0 \textit{ or } \mathbb{Z}/2 & m = 32 \\ \mathbb{Z}/2 & \text{otherwise} \end{cases}$$

The nontrivial exotic spheres in bP_{4m-2} are called Kervaire spheres.

Corollary 8.16. *For $m \neq 1, 2, 4, 8, 16, 32$, there exists a topological $(4m-2)$-manifold that cannot be given a smooth structure.*

Proof. By Theorem 8.14, there exists a manifold $M \in P_{4m-2}$ with Kervaire invariant 1. Attach a disk to M along $\partial M \simeq S^{4m-2}$ to obtain a topological manifold M'. Now M' cannot be given a smooth structure since then M' would be a smooth closed manifold with Kervaire invariant 1, contradicting Theorem 8.15. □

Kervaire [12] was the first to discover a topological manifold possessing no differentiable structure. His 10-dimensional manifold is the same as M' in the above proof when $m = 3$.

9. Summary of Results

From our results on bP_k and the Kervaire-Milnor long exact sequence, we have the following theorem.

Theorem 9.1. *Let $n \geq 4$.*

- *We have equality $\Theta_n = \operatorname{coker}(J_n)$ for*

$$\begin{cases} n \equiv 0 \mod 4 & \\ n \equiv 2 \mod 4 & n \neq 6, 14, 30, 62, 126 \\ n = 5, 13, 29, 62, 125 & \end{cases}$$

- *For $n = 6, 14, 30, 62$, we have an exact sequence*

$$0 \to \Theta_n \to \operatorname{coker}(J_n) \xrightarrow{c} \mathbb{Z}/2 \to 0$$

- *For n odd, we have an exact sequence*

$$0 \to bP_{n+1} \to \Theta_n \to \operatorname{coker}(J_n) \to 0$$

Brumfiel has shown that if $n \neq 2^k - 3$, this sequence splits ([8], [9]).

Moreover, the group bP_{4k} are cyclic of order

$$\frac{3 - (-1)^k}{2} \cdot 2^{2k-2}(2^{2k-1} - 1) \cdot \operatorname{numerator}(B_k/4k)$$

and

$$bP_{4m-2} = \begin{cases} 0 & m = 1, 2, 4, 8, 16 \\ \textit{either } 0 \textit{ or } \mathbb{Z}/2 & m = 32 \\ \mathbb{Z}/2 & \text{otherwise} \end{cases}$$

10. Examples in small dimensions

We compute the groups Θ_n for $n \leq 9$, $n \neq 3$. For $n = 1, 2$ the group Θ_n is trivial. For higher n, we will use the following computations of the stable homotopy groups of spheres and the image of J.

n	0	1	2	3	4	5	6	7	8	9
π_n^S	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$(\mathbb{Z}/2)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 3}$

By [1], the image $\operatorname{Im}(J_n) \subset \pi_n^S$ of the J-homomorphism is a cyclic group of order

$$|\operatorname{Im}(J_n)| = \begin{cases} 0 & n \equiv 2, 4, 5, 6 \mod 8 \\ 2 & n \equiv 0, 1 \mod 8 \\ \operatorname{denom}\left(\frac{B_{2s}}{4s}\right) & n = 4s - 1 \end{cases}$$

Moreover, $\operatorname{Im}(J_n)$ is a direct summand of π_n^S.

10.1. Θ_4 and Θ_5.

For $n = 4, 5$, we have $\Theta_n = \operatorname{coker}(J_n) \subset \pi_n^S = 0$.

10.2. Θ_6. We have an exact sequence

$$0 \to \Theta_6 \to \operatorname{coker}(J_6) \to \mathbb{Z}/2 \to 0$$

Since $\pi_6 SO = 0$ we have $\operatorname{coker}(J_6) = \pi_6^S$. The group π_6^s is a $\mathbb{Z}/2$ so that the short exact sequence becomes

$$0 \to \Theta_6 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \to 0$$

and therefor $\Theta_6 = 1$ is the trivial group.

10.3. Θ_7. The group Θ_7 sits in a short exact sequence

$$0 \to bP_8 \to \Theta_7 \to \operatorname{coker}(J_7) \to 0$$

The group bP_8 has order

$$|bP_8| = \frac{3 - (-1)^2}{2} \cdot 2^{4-2}(2^{4-1} - 1) \cdot \text{numerator}(B_2/8) = 28$$

The quotient group $\Theta_7/bP_8 \cong \Omega_7^{alm}$ Since $7 \not\equiv 0 \mod 4$, we have $\Omega_7^{alm} \cong \operatorname{coker} J_7$. Now

$$J_7 : \mathbb{Z} \cong \pi_7 SO \to \pi_7^S$$

has image $|\operatorname{Im} J| = \text{denominator}(B_2/8) = 240$. The stable homotopy group π_7^S also has order 240. Thus $\operatorname{coker}(J_7) = 0$ and $\Theta_7 = bP_8 = \mathbb{Z}/28$.

10.4. Θ_8. The group $\Theta_8 = \operatorname{coker}(J_8)$. The map $J_8 : \mathbb{Z}/2 = \pi_8 SO \to \pi_8^S$ is injective. The group $\pi_8^S = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ so that $\Theta_8 = \operatorname{coker}(J_8) = \mathbb{Z}/2$.

10.5. Θ_9. The group Θ_9 sits in a short exact sequence

$$0 \to bP_{10} \to \Theta_9 \to \Omega_9^{alm} \to 0$$

The group bP_{10} has order 2. The quotient group $\Theta_9/bP_{10} \cong \Omega_9^{alm}$ Since $9 \not\equiv 0 \mod 4$, we have $\Omega_9^{alm} \cong \operatorname{coker} J_9$. Now

$$J_9 : \mathbb{Z}/2 \cong \pi_9 SO \to \pi_9^S$$

is injective with image a direct summand. The stable homotopy group $\pi_9^S = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Thus $\operatorname{coker}(J_9) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ so that

$$0 \to \mathbb{Z}/2 \to \Theta_9 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to 0$$

This exact sequence splits so that $\Theta_9 = (\mathbb{Z}/2)^{\oplus 3}$ has 8 elements.

11. Constructing the Exotic 8-sphere

In §8, we constructed elements of bP_{2k} using a plumbing construction. The goal of this section is to describe a way of constructing exotic spheres that do not bound parallelizable manifolds. The first such exotic sphere occurs in dimension eight. From §10.4, we have $\Theta_8 = \operatorname{coker}(J_8) = \mathbb{Z}/2$. Let Σ^8 denote the unique nonzero element in Θ_8. We will focus on constructing Σ^8 by constructing a 9-manifold M whose boundary ∂M is Σ^8. To begin, we recall ways to construct exotic spheres in bP_n.

Motivation: For the plumbing construction described in §8 (specifically Theorem 8.14), we glued together two disk bundles of tangent bundles on spheres $\mathsf{Disk}(TS_1^n) \cup \mathsf{Disk}(TS_2^n)$. The gluing consisted of finding disks $D_i \subset S^n$ over which TS^n trivializes, $(TS^n)|_{D_i} \cong D_i \times \mathbb{D}^n$ and gluing $\mathsf{Disk}(TS_1^n) \cup \mathsf{Disk}(TS_2^n)$ along D_i by switching base and fiber, $D_1 \times \mathbb{D}^n \to D_2 \times \mathbb{D}^n$ by $(x, y) \mapsto (y, x)$. Under certain conditions, the resulting manifold had boundary an exotic sphere. The plumbing used in Theorem 8.4 was more general: we glued together 8 copies of $\mathsf{Disk}(TS^n)$ according to a matrix E_8. Think of plumbing as "gluing tangent bundles of spheres together with a twist."
Generalization: We will construct exotic sphere outside of bP_n by a construction like plumbing but

using bundles on spheres other than the tangent bundle. Think "gluing weird bundles of spheres together with a twist."

Remark. Let M be the result of plumbing. Then TM is trivial and hence ∂M bounds a parallelizable manifold. If we want to construct exotic spheres that do not lie in bP_n using a similar construction, we need our modified plumbing construction to result in a manifold that is not parallelizable. From Theorem 8.4, the proof that TM was trivial crucially used the fact that tangent bundles of spheres are stable trivial. In our modified plumbing construction we therefore want to glue together disk bundles that are *not* stably trivial.

Definition. Let $n < m$. Let $f \in \pi_{n-1}SO(m)$ and $g \in \pi_{m-1}SO(n)$ with corresponding vector bundles $\xi \downarrow S^n$ and $\mu \downarrow S^m$ on S^n and S^m, respectively. Let W by the $(m+n)$-manifold obtained by plumbing together $\mathsf{Disk}(\xi)$ and $\mathsf{Disk}(\mu)$. The Milnor pairing is defined to be the map

$$M : \pi_{n-1}SO(m) \times \pi_{m-1}SO(n) \to \Theta_{n+m-1}$$

sending (f, g) to the boundary ∂W.

Remark. Although we are referring to the Milnor pairing as a "generalization" of the plumbing construction, historically the Milnor pairing was defined first. In his 1959 paper [18], Milnor defines the pairing. The special case using tangent bundles of spheres is discussed in more detail in Browder's book [7].

Example. Let $\tau_n \in \pi_{n-1}SO(n)$ be the transition function for the tangent bundle of the n-sphere. The standard plumbing construction can be recovered from the Milnor pairing as $M(\tau_n, \tau_n)$. Note that the condition $n < m$ in the definition of the Milnor pairing is not satisfied here. The condition $n < m$ is necessary to prove that $M(f, g)$ will be a homotopy sphere. When plumbing tangent bundles of spheres, an alternative argument can be made (c.f., Theorem 8.4).

There is a claim being made in the definition of the Milnor pairing: that $M(f, g)$ is a homotopy sphere. We prove this.

Proposition 11.1. *If $n < m$ and $f \in \pi_{n-1}SO(m)$, $g \in \pi_{m-1}SO(n)$, then $M(f, g)$ is homeomorphic to S^{n+m-1}.*

The proof given here will follow [18, Lem. 1,3].

To prove this proposition, it helps to have a direct construction of $M(f, g)$ not as a boundary.

Lemma 11.2. *Let f, g as in the statement of Proposition 11.1. Let D^k_- and D^k_+ denote the southern and northern hemispheres of S^k. Let $S^{k-1}_- = \partial D^k_-$ and $S^{k-1}_+ = \partial D^k_+$ be their respective boundaries. The manifold $M(f, g)$ is homeomorphic to*

$$(D^n_+ \times S^{m-1}) \cup_\theta (D^m_- \times S^{n-1})$$

where $\theta : S^{n-1}_+ \times S^{m-1} \to S^{m-1}_- \times S^{n-1}$ by

$$\theta(a, b) = (f_a(b), g_{f_a(b)}(a))$$

Proof. The manifold $M(f, g)$ is the boundary of the $(n+m)$-manifold W obtained by plumbing together the disk bundles corresponding to f and g. Let $\xi \downarrow S^n$ be the rank m vector bundle on S^n with transition function f. Let $\mu \downarrow S^m$ be the rank n vector bundle on S^m with transition function g. Then

$$\mathsf{Disk}(\xi) = (D^n_+ \times \mathbb{D}^m) \cup_F (D^n_- \times \mathbb{D}^m)$$

where $F : S^{n-1}_+ \times \mathbb{D}^m \to S^{n-1}_- \times \mathbb{D}^m$ by $F(a, b) = (a, f_a(b))$. Similarly,

$$\mathsf{Disk}(\mu) = (D^m_+ \times \mathbb{D}^n) \cup_G (D^m_- \times \mathbb{D}^n)$$

where $G : S^{m-1}_+ \times \mathbb{D}^m \to S^{m-1}_- \times \mathbb{D}^n$ by $G(a,b) = (a, g_a(b))$. The plumbed manifold W is then

$$W = \mathsf{Disk}(\xi) \cup \mathsf{Disk}(\mu)$$

glued along the isomorphism $\chi : D^n_- \times \mathbb{D}^m \to D^m_+ \times \mathbb{D}^n$ sending $(x,y) \mapsto (y,x)$. We have defined $M(f,g)$ to be the boundary ∂W. We have

$$\begin{aligned}
\partial W &= \partial\Big(\mathsf{Disk}(\xi) \cup \mathsf{Disk}(\mu)\Big) \\
&= \partial\Big((D^n_+ \times \mathbb{D}^m) \cup_F (D^n_- \times \mathbb{D}^m) \bigcup (D^m_+ \times \mathbb{D}^n) \cup_G (D^m_- \times \mathbb{D}^n)\Big) \\
&= (D^n_+ \times S^{m-1}) \cup_\theta (D^m_- \times S^{n-1})
\end{aligned}$$

where $\theta : S^{n-1}_+ \times S^{m-1} \to S^{m-1}_- \times S^{n-1}$ by the composition $G \circ \chi \circ F$; i.e., θ sends (a,b) to

$$(a,b) \xmapsto{F} (a, f_a(b)) \xmapsto{\chi} (f_a(b), a) \xmapsto{G} (f_a(b), g_{f_a(b)}(a))$$

$\square$

For $x \in S^{m-1}$ and $y \in S^{n-1}$ set $\theta(x,y) = (x', y')$. Note that the manifold $(D^m \times S^{n-1}) \cup_\theta (S^{m-1} \times D^n)$ is homeomorphic to the manifold obtained by gluing

$$(\mathbb{R}^m \times S^{n-1}) \cup (S^{m-1} \times \mathbb{R}^n)$$

along $\hat\theta : (\mathbb{R}^m \setminus \{0\}) \times S^{n-1} \to S^{m-1} \times (\mathbb{R}^n \setminus \{0\})$ by

$$\hat\theta(tx, y) = (x', \frac{1}{t} y')$$

for $x \in S^{m-1}$, $y \in S^{n-1}$, and $0 < t < \infty$.

Remark. When θ is the identity, we recover the standard sphere S^{m+n-1} since

$$S^{m+n-1} = \partial(\mathbb{D}^m \times \mathbb{D}^n) = (S^{m-1} \times \mathbb{D}^n) \cup_{\mathrm{Id}} (\mathbb{D}^m \times S^{n-1})$$

Every homotopy sphere has a Morse function with exactly two critical points. For the standard sphere embedded in Euclidean space, one can define such a Morse function by the height function. In terms of the description of S^{m+n-1} using $\hat\theta$ (where $\theta = \mathrm{Id}$), the height function is given by $H : S^{m+n-1} \to [-1,1]$ with

$$H(x,y) = \frac{h(y)}{\sqrt{1+t^2}} = \frac{\frac{1}{t} h(y')}{\sqrt{1 + \frac{1}{t^2}}}$$

Proof of Proposition 11.1. From the long exact sequence in homotopy for the fibration

$$SO(m-1) \xrightarrow{i} SO(m) \to S^{m-1}$$

we see that i induces a surjection,

$$\pi_{n-1} SO(m-1) \xrightarrow{i_*} \pi_{n-1} SO(m) \to \pi_{n-1} S^{m-1} = 0$$

We may therefore assume $f \in \pi_{n-1} SO(m)$ is in the image of i_*. Say $f = i_*(f')$.

We will construct a Morse function $h : M(f,g) \to \mathbb{R}$ with exactly two critical points. By [21, Thm. 2], this will imply that $M(f,g)$ is a homeomorphic to S^{n+m-1}.

By Lemma 11.2, and the above remarks, we can describe M as

$$(\mathbb{R}^m \times S^{n-1}) \cup_{\hat\theta} (S^{m-1} \times \mathbb{R}^n)$$

Note that since $f' : S^{n-1} \to SO(m-1)$, for each $x \in S^{n-1}$ the map $(i \circ f')_x$ acts on $\mathbb{R}^m$ by rotating $\mathbb{R}^{m-1} \times \{0\} \subset \mathbb{R}^m$. Put coordinates $y = (y_1, \ldots, y_m)$ on $\mathbb{R}^m$ and define $h(y) = y_m$ to be the height function. Then $h(y) = h\big(i \circ f')_x(y)\big)$ since $(i \circ f')_x$ fixes the mth coordinate. We can therefore use the same Morse function as we described for the standard sphere in the above remark, $H : M \to [-1,1]$ by

$$H(x,y) = \frac{h(y)}{\sqrt{1+t^2}} = \frac{\frac{1}{t}h(y')}{\sqrt{1+\frac{1}{t^2}}}$$

Since $h(y) = h(y')$, the map H is well-defined. One can check that H has exactly two nondegenerate critical points. Hence $M(f,g)$ is homeomorphic to S^{m+n-1}. $\square$

We want an exotic 8-sphere. Set $n+m-1 = 8$. If $n < m$, we are restricted to the following cases of the Milnor pairing:

(n,m)	$\pi_{n-1}SO(m) \times \pi_{m-1}SO(n)$	Result
(1,8)	$\pi_0 SO(8) \times \pi_7 SO(1)$	0×0
(2,7)	$\pi_1 SO(7) \times \pi_6 SO(2)$	$\mathbb{Z}/2 \times 0$
(3,6)	$\pi_2 SO(6) \times \pi_5 SO(3)$	$0 \times \mathbb{Z}/2$
(4,5)	$\pi_3 SO(5) \times \pi_4 SO(4)$	$\mathbb{Z} \times (\mathbb{Z}/2 \oplus \mathbb{Z}/2)$

The groups $\pi_{n-1}SO(m)$ for $n = 1, \ldots, 4$ listed above are all stable. We know the stable homotopy groups of SO by Bott periodicity. The values of $\pi_{m-1}SO(n)$ for $n = 1, \ldots, 4$ listed above can be computed using the identifications $SO(1) \cong \{\text{pt}\}$, $SO(2) \cong S^1$, $SO(3) \cong \mathbb{RP}^3$, and the fact that $S^3 \times S^3$ double covers $SO(4)$.

The only choice of (n,m) for which the Milnor pairing can take in two nontrivial bundles is $(4,5)$. We therefore consider the pairing

$$M : \pi_3 SO(5) \times \pi_4 SO(4) \to \Theta_8$$

Note that $\pi_3 SO(5)$ is stable so that a nontrivial element in $f \in \pi_3 SO(5)$ will correspond to a stably nontrivial vector bundle on S^4. We therefore have hope of constructing a non-parallelizable manifold.

Theorem 11.3. *Write $\eta_3\tau_4$ for the composite*

$$S^4 \xrightarrow{\eta_3} S^3 \xrightarrow{\tau_4} SO(4)$$

where η_3 is the unique essential map $S^4 \to S^3$ and τ_4 is the transition function for the tangent bundle TS^4. Let $\alpha \in \pi_3 SO(5) \cong \mathbb{Z}$ be the generator $+1$. Then $M(\alpha, \eta_3\tau_4) = \Sigma^8$ is the exotic 8-sphere.

The map $F : \Theta_8 \to \text{coker}(J_8)$ from §5 is an isomorphism. To see that $M(\alpha, \eta_3\tau_4)$ is nontrivial in Θ_8 it therefore suffices to show that $F(M(\alpha, \eta_3\tau_4))$ is nonzero in $\text{coker}(J_8)$. Recall from §10.4 that $\pi_8^S = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ with one copy of $\mathbb{Z}/2$ corresponding to the image of J_8. Let $\sigma\eta \in \pi_8^S$ denote the generator of the image of J_8 and $\epsilon \in \pi_8^S$ denote the generator of the other copy of $\mathbb{Z}/2$. Set $\bar{\nu} = \epsilon + \sigma\eta$. Then $\text{coker}(J_8) = \mathbb{Z}/2$ is generated by the class of $[\epsilon] = [\bar{\nu}]$.

Theorem 11.4 (Smith, Steer, Wood)**.** *Under the Pontryagin-Thom isomorphism $\pi_8^S \cong \Omega_8^{fr}$, the element $\bar{\nu}$ corresponds to the class $[SU(3), \mathcal{L}]$ of the manifold $SU(3)$ with its left-invariant Lie group framing $\mathcal{L}$.*

We will show that $F(M(\alpha, \eta_3\tau_4))$ is nonzero by showing that the manifold $M(\alpha, \eta_3\tau_4)$ is framed cobordant to $[SU(3), \mathcal{L}]$. Note that by the proof of Theorem 4.1, trivializations of $TM(\alpha, \eta_3\tau_3) \oplus \epsilon^1$ give framed manifolds in Ω_8^{fr} that differ by things in the image of $\bar{J}$. Since we are considering everything mod the image of J, we can take any trivialization of $TM(\alpha, \eta_3\tau_4) \oplus \epsilon^1$.

Theorem 11.5. *There is a framing Ψ on $M(\alpha, \eta_3\tau_4)$ coming from a trivialization of $TM(\alpha, \eta_3\tau_4) \oplus \epsilon^1$ so that $[M(\alpha, \eta_3\tau_4), \Psi] \in \Omega_8^{fr}$ is framed cobordant to $[SU(3), \mathcal{L}]$.*

We begin by identifying $\eta_3\tau_4$ as the transition function for $SU(3)$. Note that $SU(3)$ is the total space of a principal $SU(2)$-bundle,

$$SU(2) \to SU(3) \to S^5$$

Let ξ be the corresponding rank 2 complex vector bundle on S^5 with total space $E(\xi)$,

$$\mathbb{C}^2 \to E(\xi) \to S^5$$

The transition function for ξ is a map $f_\xi : S^4 \to SU(2)$. We have a map $r : SU(2) \to SO(4)$.

Lemma 11.6. *As elements of $\pi_4 SO(4)$, we have $r \circ f_\xi = \eta_3\tau_4$.*

Proof. Note that the tangent bundle of S^4 has an almost complex structure so that we have a lift:

$$\begin{tikzcd}
 & & SU(2) \arrow[d, "r"] \\
S^4 \arrow[r, "\eta_3"'] & S^3 \arrow[ur, dashed] \arrow[r, "\tau_4"'] & SO(4)
\end{tikzcd}$$

Thus $\eta_3\tau_4 \in \pi_4 SO(4)$ is in the image of $r_* : \pi_4 SU(2) \to \pi_4 SO(4)$. Say $\eta_3\tau_4 = r_*(g)$. Then g and f_ξ are both nontrivial elements of $\pi_4 SU(2)$. From the long exact sequence of the fibration

$$S^1 \cong SU(1) \to SU(2) \to S^3$$

we have

$$0 = \pi_4 SU(1) \to \pi_4 SU(2) \to \pi_4 S^3 \to \pi_3 S^1 = 0$$

Thus $\pi_4 SU(3) = \pi_4 S^3 = \mathbb{Z}/2$. Hence we must have $g = f_\xi$ and therefore $\eta_3\tau_4 = r \circ f_\xi$ (up to homotopy). $\square$

Set $f = \eta_3\tau_4$. The above lemma then states that f is homotopic to the composition

$$f : S^4 \xrightarrow{f_\xi} SU(2) \xrightarrow{r} SO(4)$$

We need a result describing the trivialization $\mathcal{L}$ of $TSU(3)$ more precisely. Let $i : S^3 \to SU(3)$ be the inclusion of a fiber. Then pulling back the trivialization of $TSU(3)$ by $\mathcal{L}$ to S^3 gives a stable trivialization $i^*\mathcal{L}$ of TS^3. Comparing this trivialization to the standard stable trivialization of TS^3 coming from the embedding $S^3 \subset \mathbb{R}^4$ gives an element of $\pi_3 SO$. Let α be this element. For computations of this element, see [28, Lem. 3.3] and [29, Appendix].

The following proof is following Smith, [28, Thm. 3.2].

Proof of Theorem 11.5. We perform framed surgery on the fiber $S^3 \cong SU(2) \to SU(3)$ until we obtain $M(\alpha, f)$.

We begin by describing $SU(3)$ and $M(\alpha, f)$ more explicitly. Let D^5_+ and D^5_- be the northern and southern hemispheres of S^5. As a sphere bundle, we have

$$SU(3) \cong (D^5_+ \times S^3) \cup_\kappa (D^5_- \times S^3)$$

where $\kappa : S^4_+ \times S^3 \to S^4_- \times S^3$ by

$$\kappa(x, y) = (x, f_\xi(x)(y)) = (x, f_x(y))$$

By Lemma 11.2, the manifold $M(\alpha, f)$ is

$$(D^4_+ \times S^4) \cup_\theta (D^5_- \times S^3)$$

where $\theta : S^3_+ \times S^4 \to S^4_- \times S^3$ by

$$\theta(u, v) = (\alpha_u(v), f_{\alpha_u(v)}(u))$$

To perform surgery on $S^3 \to SU(3)$ we need an embedding $\varphi : S^3 \times \mathbb{D}^5 \to SU(3)$. Take φ to be the identification $S^3 \times \mathbb{D}^5 \cong S^3 \times D^5_+ \to SU(3)$. Let $\varphi_\alpha : S^3 \times \mathbb{D}^4 \to SU(3)$ by

$$\varphi_\alpha(x, y) = \varphi(x, \alpha_x(y))$$

The manifold obtained by doing surgery to $SU(3)$ using φ_α is

$$\begin{aligned}
\chi(SU(3), \varphi_\alpha) &= \Big(SU(3) \setminus \varphi_\alpha(S^3 \times \{0\})\Big) \cup (\mathbb{D}^4 \times S^4) \\
&= \Big(\big((D^5_+ \setminus \{0\}) \times S^3\big) \cup_\kappa (D^5_- \times S^3)\Big) \cup (\mathbb{D}^4 \times S^4) \\
&= (\mathbb{D}^4 \times S^4) \cup (D^5_- \times S^3)
\end{aligned}$$

where $\varphi_\alpha(u, tv) = \varphi(u, \alpha_u(tv))$ is glued to (tu, v) for every $u \in S^3$, $v \in S^4$ and $t \in (0, 1]$. Now the surgery glues an element $(u, v) \in S^3 \times S^4 \subset \mathbb{D}^4 \times S^4$ to $(\alpha_u(v), u) \in S^4_+ \times S^3$. Inside $SU(3)$, the element $(\alpha_u(v), u)$ is glued using κ to $(\alpha_u(v), f_{\alpha_u(v)}(u))$ in $S^4_- \times S^3$. But now

$$(u, v) \sim (\alpha_u(v), f_{\alpha_u(v)}(u))$$

is exactly the identification made in $M(\alpha, f)$. Thus we can perform a single surgery on $SU(3)$ to obtain $M(\alpha, f)$. Hence $SU(3)$ and $M(\alpha, f)$ are cobordant. To see that $SU(3)$ and $M(\alpha, f)$ are *framed* cobordant, we need to show that the surgery $\chi(SU(3), \varphi_\alpha)$ can be framed. By the claim made in Proposition 6.4, the obstruction to framing the surgery using φ_α is

$$\gamma(\varphi_\alpha) = \gamma(\varphi) + s_*(\alpha)$$

where $s_* : \pi_3 SO(5) \to \pi_3 SO(9)$ induced by inclusion. Since $\pi_3 SO(5)$ and $\pi_3 SO(9)$ are both stable, s_* is the identity. By the remark after the proof of Proposition 6.4, the obstruction $\gamma(\varphi)$ is the homotopy class $\gamma(\varphi) \in \pi_3 SO$ obtained by restricting the trivialization of $TSU(3)$ to the fiber $\varphi(S^3 \times \{0\})$ over the north pole. By definition, the trivialization $\mathcal{L}$ of $TSU(3)$ restricted to a fiber is $i^*\beta$ is α. Thus

$$\gamma(\varphi_\alpha) = \gamma(\varphi) + s_*\alpha = i^*\beta - i^*\beta = 0$$

The surgery $\chi(SU(3), \varphi_\alpha)$ can therefore be framed. Hence $[SU(3); \mathcal{L}]$ is framed cobordant to $M(\alpha, \eta_3\tau_4)$ proving the theorem. □

This concludes the proof of Theorem 11.5. Together with Theorem 11.4, this implies that $M(\alpha, \eta_3\tau_4)$ is the exotic 8-sphere.

Remark. The proof given above is rather roundabout. We showed that $M(\alpha, \eta_3\tau_4)$ produced the exotic 8-sphere by showing it was framed cobordant to $[SU(3); \mathcal{L}]$. We then appealed to a theorem computing $[SU(3); \mathcal{L}]$ in $\mathrm{coker}(J)$. A more direct proof that $M(\alpha, \eta_3\tau_4) \in \Theta_8$ is nonzero is given in [10, Example 1]. Recall that $M(\alpha, \eta_3\tau_4)$ was constructed as the boundary of a plumbed manifold W. In [10], an invariant of of $\Delta(W) \subset \mathrm{coker}(J)$ of W is defined. Frank is able to compute this invariant for $M(\alpha, \eta_3\tau_4)$ and thus conclude that $M(\alpha, \eta_3\tau_4)$ is the nonzero element of Θ_8.

12. Appendix

12.1. **Bott Periodicity.** We discuss both real and complex Bott periodicity. For the 2-fold periodicity of the unitary group, we follow the Morse theoretic proof given in [20]. For the 8-fold periodicity of the orthogonal group, we list the Morse theoretic description of each loop space $\Omega^i O$, $i = 1, \dots, 8$ without proof. The results are as follows,

Theorem 12.1 (Bott, [2])**.** *The homotopy groups of the stable unitary group U are 2-periodic and the homotopy group of the stable orthogonal group O are 8-periodic.*

The proof proceeds by defining a highly connected map $B : Gr_m(\mathbb{C}^{2m}) \to \Omega SU(2m)$ and deducing that the composition

$$\pi_k U = \pi_k BU = \pi_{k+1} Gr_m(\mathbb{C}^{2m}) \xrightarrow{f} \pi_{k+1}\Omega SU(2m) = \pi_{k+2} SU(2m)$$

is an isomorphism for m large.

We can reformulate Bott periodicity using loop spaces,

Theorem 12.2. *There are a homotopy equivalences $U \simeq \Omega^2 U$ and $O \simeq \Omega^8 O$.*

Let $M = U$ or O. The proof will involve using Morse theoretic techniques on the loop space ΩM. Since ΩM is not a finite-dimensional manifold, our first step is to replace ΩM with a good finite approximation. We will approximate ΩM with $\Omega^{\min}_{x,y} M$, the space of minimal geodesics between $x, y \in U$. The space of minimal geodesics is a "good" approximation of ΩM in the sense that we can compute the connectivity of the inclusion $\Omega^{\min}_{x,y} M \subset \Omega_{x,y} M$. The proof is broken into four main parts:

- Part 1: It is shown that the connectivity of the inclusion $\Omega^{\min}_{x,y} M \subset \Omega_{x,y} M$ depends on an invariant of the geodesics.
- Part 2: The invariant is computed for geodesics on $U(m)$ with m large.
- Part 3: The space $\Omega^{\min}_{x,y} SU(2m)$ is identified with the Grassmannian $Gr_m \mathbb{C}^{2m}$. The map $B : Gr_m(\mathbb{C}^{2m}) \to \Omega SU(2m)$ discussed above is therefore inclusion of minimal geodesics.
- Part 4: We state, without proof, the computations for $O(n)$, n large.

Applying the results in Part 1 to the computations in Part 2 and 3 will prove Bott periodicity.

For $x, y \in M$, let $\Omega_{x,y} M$ denote the space of piecewise smooth paths $f : [0,1] \to M$ so that $f(0) = x$ and $f(1) = y$. To state the result of Part 1, we need some definitions. In the spirit of Morse theory, we will define a function $E : \Omega M \to \mathbb{R}$ so that geodesics are critical points of E and the connectivity of $\Omega^{\min}_{x,y} M \subset \Omega_{x,y} M$ depends on the index of the critical points of E.

Definition. Let $E : \Omega_{x,y} M \to \mathbb{R}$ be the energy function defined on a path $\omega \in \Omega_{x,y} M$ as

$$E(\omega) = \int_0^1 \left\| \frac{d\omega}{dt} \right\|^2 dt$$

To talk about critical points of E, we need to understand the derivative of E. Since $\Omega_{x,y} M$ is not a manifold, we need to define $T_\omega \Omega_{x,y} M$ and dE.

Definition. Let ω an element of $\Omega_{x,y} M$. Define the tangent space of $\Omega_{x,y}$ at ω to be the vector space of piecewise smooth vector fields along ω that start and end at 0.

We define what it means to be a critical point of E using calculus of variations.

Definition. A variation of ω is a function $\bar{\alpha} : (-\epsilon . \epsilon) \to \Omega_{x,y} M$ for some $\epsilon > 0$ so that
(1) $\bar{\alpha}(0) = \omega$ and
(2) $\alpha : (-\epsilon, \epsilon) \times [0,1] \to M$ by $\alpha(u,t) = \bar{\alpha}(u)(t)$ is piecewise smooth on some subdivision of $[0,1]$.
Similarly, define a 2-parameter variation of ω and vector fields $W_1, W_2 \in T_\omega \Omega_{x,y} M$ to be a map $\alpha : U \times [0,1] \to M$ with $U \subset \mathbb{R}^2$ a neighborhood of $(0,0)$ and such that
(1) $\alpha(0,0,t) = \omega(t)$ and
(2) $\frac{\partial \alpha}{\partial u_i}(0,0,t) = W_i(t)$, $i = 1, 2$.
Set $\bar{\alpha}(u_1, u_2)(t) = \alpha(u_1, u_2, t)$.

Now $E \circ \bar{\alpha} : (-\epsilon, \epsilon) \to \mathbb{R}$ has a derivative.

Definition. A path ω is a critical point of E if $\frac{d}{du} E(\bar{\alpha}(u))\Big|_{u=0} = 0$ for every variation $\bar{\alpha}$ of ω.

The Hessian of E at a critical point ω is

$$E_{**} = \frac{\partial^2}{\partial u_1 \partial u_2} E(\bar{\alpha}(u_1, u_2))\Big|_{(0,0)}$$

Claim. *A path $\omega \in \Omega_{x,y}M$ is a critical point of E if and only if ω is a geodesic.*

For a proof see [20, Cor. 12.3].

As usual, the Hessian $E_{**} : T_\omega \Omega_{x,y}M \times T_\omega \Omega_{x,y}M \to \mathbb{R}$ is bilinear. The maximum dimension of a subspace of $T_\omega \Omega_{x,y}M$ on which E_{**} is negative definite is called the index of E_{**}.

Claim. *The minimal geodesics on M correspond to minimums of E. In particular, if γ is a minimal geodesics on M, then $E_{**}(\gamma)$ has index 0.*

For Part 1, the main result is as follows,

Theorem 12.3. *If the subspace $\Omega^{\min}_{x,y}M \subset \Omega_{x,y}M$ of minimal geodesics is a topological manifold and the smallest index of a non-minimal critical point of E is equal to k, then the inclusion $\Omega^{\min}_{x,y}M \to \Omega_{x,y}M$ is $(k-1)$-connected.*

Complex Case:

Theorem 12.4. *Every non-minimal geodesic from I to $-I$ in $SU(2m)$ has index $\geq 2m+2$.*

We mentioned above that the connectivity of $\Omega^{\min}_{x,y}M \subset \Omega_{x,y}M$ was computable; however, the computation is difficult and involves many pages of work. The work includes both Theorems 15.1 and 20.5 of [20]. The completion of the proof of this computation can be found in [20, Lem. 23.2].

Lemma 12.5. *The spaces $\Omega^{\min}_{I,-I}SU(2m)$ and $Gr_m(\mathbb{C}^{2m})$ are homeomorphic.*

Proof. We describe the map. Details can be found in [20, Lem. 23.1]. The space $T_I SU(2m)$ can be identified with traceless, skew-symmetric $2m \times 2m$ matrices. Such a matrix $A \in T_i SU(2m)$ corresponds to a geodesic from I to $-I$ if $\exp(A) = -I$. Since this property is invariant under conjugation, we can assume A is diagonal, $A = \mathrm{diag}(a_1, \ldots, a_{2m})$. Under these restrictions on A, we must have $a_r = k_r i\pi$ for $k_1, \ldots, k_{2m}$ odd integers that sum to zero. The geodesic $\exp(tA)$ is minimal if and only if $k_1, \ldots, k_{2m} \in \{1, -1\}$. Such matrices are completely determined by their positive eigenspace. The map $\Omega^{\min}_{I,-I}SU(2m) \to Gr_m(\mathbb{C}^m)$ sending such a matrix A to its positive eigenspace defines the homeomorphism. □

Real Case: For the real case, let $\Omega^{\min}_{I,-I}O(n)$ denote the space of minimal geodesics in $O(n)$ from I to $-I$. For later computations, we need n to be divisible by a high power of 2. For now, it suffices for $n = 2m$ to be even. The analogue of Theorem 12.4 is the following,

Lemma 12.6. *Any non-minimal geodesic from I to $-I$ in $O(2m)$ has index $\geq 2m-2$.*

Corollary 12.7. *By Theorem 12.3, the inclusion $\Omega^{\min}_{I,-I}O(n) \to \Omega O(n)$ induces isomorphisms*

$$\pi_i \Omega^{\min}_{I,-I}O(n) \simeq \pi_{i+1}O(n)$$

for $i \leq n-4$.

The space $\Omega^{\min}_{I,-I}O(n)$ can be described as the space of all complex structures on $\mathbb{R}^n$.

Definition. A complex structure J on $\mathbb{R}^n$ is a linear transformation $J : \mathbb{R}^n \to \mathbb{R}^n$, belonging to $O(n)$, so that $J^2 = -I$.

Let $\Omega_1(n)$ denote the space of all complex structures on $\mathbb{R}^n$.

Lemma 12.8. *The spaces $\Omega^{\min}_{I,-I}O(n)$ and $\Omega_1(n)$ are homeomorphic. Moreover, for n even we have an isomorphism $\Omega_1(n) \cong O(n)/U(n/2)$ where $U(n/2) \subset O(n)$ is the subspace of orthogonal transformations that commute with a fixed complex structure J_1.*

Fix $k-1$ complex structures $J_1, \dots, J_{k-1} \in \Omega_1(n)$ such that $J_rJ_s + J_sJ_r = 0$ for $r \neq s$.

Definition. Let $\Omega_k(n)$ denote the set of all complex structures on $\mathbb{R}^n$ which anti-commute with $J_1, \dots, J_{k-1}$.

Choose $J_1, \dots, J_{k-1}$ so that $\Omega_k(n)$ is nonempty. These sets define a filtration of $O(n)$:

$$\Omega_k(n) \subset \Omega_{k-1}(n) \subset \cdots \subset \Omega_1(n) \subset O(n)$$

Set $\Omega_0(n) = O(n)$ and define stable versions

$$\Omega_k = \lim_{n\to\infty} \Omega_k(n)$$

and $O = \Omega_0$.

Theorem 12.9. *For each $k \geq 0$, the stable map $\Omega_{k+1} \to \Omega\Omega_k$ induced by the inclusion $\Omega_{k+1}(n) \to \Omega\Omega_k(n)$ is a homotopy equivalence.*

Thus

$$\pi_k O \simeq \pi_{k-1}\Omega_1 \simeq \pi_{k-2}\Omega_2 \simeq \cdots \simeq \pi_1\Omega_{n-1}$$

One can identify the loop spaces $\Omega_i(n)$ for $i = 1, \dots 8$ and $n = 16r$ large as follows

$$\begin{aligned}
\Omega_0(n) &\cong O(16r)\\
\Omega_1(n) &\cong O(16r)/U(8r)\\
\Omega_2(n) &\cong U(8r)/Sp(4r)\\
\Omega_3(n) &\cong Gr(\mathbb{H}^{4r})\\
\Omega_4(n) &\cong Sp(2r)\\
\Omega_5(n) &\cong Sp(2r)/U(2r)\\
\Omega_6(n) &\cong U(2r)/O(2r)\\
\Omega_7(n) &\cong Gr(\mathbb{R}^{2r})\\
\Omega_8(n) &\cong O(r)
\end{aligned}$$

where $Gr(V)$ is the Grasssman manifold of all subspaces of a vector space V. Computing the fundamental groups of each of these spaces gives the following table,

$n \mod 8$	0	1	2	3	4	5	6	7
$\pi_n(SO)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

12.2. **Characteristic Classes.** We give a brief overview of characteristic classes and their properties.

Characteristic classes are invariants of vector bundles that live in the cohomology of the base; e.g., if $\xi : V \to B$ is a vector bundle and c is a characteristic class, then $c(\xi) \in H^*(B)$. We define characteristic classes by choosing specific universal classes and pulling them back along classifying maps. Our first step is transferring from vector bundles to principal bundles.

Claim. *Isomorphism classes of rank n vector bundles are in bijective correspondence with isomorphism classes of principal $O(n)$-bundles.*

Moreover, the bijection preserves various things one can do to vector bundles such as direct sum and pullbacks. If our vector bundle started off with extra structure, the corresponding principal bundle will have the same extra structure. For example, if we start with an oriented vector bundle, the corresponding principal bundle will be an $SO(n)$-bundle. Next we use the fact that principal G-bundles are classified by a universal G-bundle.

Claim. *For any topological group G, there exists a principal G-bundle $EG \to BG$ so that every other principal G-bundle $E \to B$ can be realized as a pullback*

$$\begin{tikzcd} E \cong \sigma^*(EG) \arrow[r] \arrow[d] & EG \arrow[d] \\ B \arrow[r, "\sigma"] & BG \end{tikzcd}$$

along some map $\sigma : B \to BG$. The map σ is unique up to homotopy.

The map σ is called the classifying map of the principal bundle $E \to B$. The classifying map of a vector bundle will mean the classifying map of the corresponding principal bundle. We therefore have an isomorphism $\mathrm{Vect}_n(B) \cong [B, BO(n)]$ between rank n vector bundles on B and homotopy classes of maps from B into $BO(n)$. Similarly, we have isomorphism between vector bundles on B with G structure and $[B, BG]$.

Definition. Fix $c \in H^*(BG)$. Let ξ be a vector bundle with G structure and classifying map $\sigma : B \to BG$. The c characteristic class of ξ is the cohomology class $c(\xi) := \sigma^*(c) \in H^*(B)$.

We get the well-known characteristic classes (e.g., Chern classes, Stiefel-Whitney classes) by computing cohomology rings $H^*(BG)$ for specific G.

Example. The cohomology ring $H^*(BO(n); \mathbb{Z}/2)$ is a truncated polynomial ring with generators w_i in degree i for each $i = 1, \dots, n$ where everything in degree $> n$ is zero. The w_i characteristic classes given by pulling back w_i along classifying maps are called Stiefel-Whitney classes. The total Stiefel-Whitney class is $w = 1 + w_1 + \cdots + w_n$.

Example. The cohomology ring $H^*(BU(n); \mathbb{Z})$ is a truncated polynomial ring with generators c_i in degree $2i$ for each $i = 1, \dots, \lfloor n/2 \rfloor$ where everything in degree $> n$ is zero. The c_i characteristic classes given by pulling back c_i along classifying maps are called Chern classes. The total Chern class is $c = 1 + c_1 + \cdots + c_{\lfloor n/2 \rfloor}$.

Example. The cohomology ring $H^*(BSO(n); \mathbb{Z})$ is a truncated polynomial ring with generators p_i in degree $4i$ for each $i = 1, \dots, \lfloor n/4 \rfloor$ where everything in degree $> n$ is zero. The p_i characteristic classes given by pulling back p_i along classifying maps are called Pontryagin classes. Alternatively, if ξ is a rank n vector bundle on B with $SO(n)$-structure, then we can complexify ξ to obtain a bundle ξ^C on B with $U(n)$-structure. The Pontryagin classes of ξ are the Chern classes of ξ^C, $p_i(\xi) = c_{2i}(\xi^C)$. The total Pontryagin class is $p = 1 + p_1 + \cdots + p_{\lfloor n/4 \rfloor}$.

Characteristic classes satisfy the following properties,

Proposition 12.10. *Let $c \in H^*(BG)$ a characteristic class. Let ξ, η vector bundles on B with G-structure and classifying maps $\sigma, \tau : B \to BG$.*

(1) If ξ is the trivial bundle, then $c(\xi) = 0$.

(2) Characteristic classes are natural.

(3) If ξ, η are vector bundles over the same space, then the total Stiefel-Whitney class of $\xi \oplus \eta$ can be expressed in terms of the total classes of ξ and η,

$$w(\xi \oplus \eta) = w(\xi)w(\eta)$$

and similarly for the total Pontryagin class and the total Chern class.

12.3. **Obstruction Theory and Characteristic Classes.** The goal of this section is to interpret certain Pontryagin classes as obstruction classes. A good reference for this material is [22, §12]. In particular, we want to show that $p_k(M) = h_* q_* \mathfrak{o}_n(M)$ where notation is as in Lemma 3.10. To do this, we need a third, more general, definition of obstruction classes (cf. §3.1).

Note that $T\Sigma \oplus \epsilon^1$ is trivial if and only if the corresponding frame bundle has a global section. We can always define a section of the frame bundle on the 0-skeleton of Σ. Our first step is to describe the obstruction to extending sections to higher skeletons.

Lemma 12.11. *Let B be a simply connected manifold and $F \to E \xrightarrow{p} B$ a fibration. Let $s : B^n \to E$ a section over the n-skeleton of B. There is a well-defined obstruction class $\sigma_n \in H^{n+1}(B, \pi_n F)$ for extending s to a section over the $(n+1)$-skeleton of B.*

Proof. We define a cochain $\sigma_n \in C^{n+1}(B, \pi_n F) = \mathrm{Hom}(C_{n+1}(B), \pi_n F)$. Let Δ be an $(n+1)$ simplex in B. Then $\partial\Delta \subset B^n$, so we have a map $s|_{\partial\Delta} : \partial\Delta \cong S^n \to E$. Since s is a section, the map $s|_{\partial\Delta}$ takes S^n into $p^{-1}(\Delta)$,

$$s|_{\partial\Delta} : S^n \to p^{-1}(\Delta).$$

Since Δ is contractible, $p^{-1}(\Delta)$ is homotopy equivalent to the fiber F. The section s_n therefore defines a map from $(n+1)$-simplices Δ to homotopy classes of maps $S^n \to F$, i.e., a cochain $\sigma_n \in C^{n+1}(B, \pi_n F)$. The cochain σ_n gives a well-defined cohomology class $\sigma_n \in H^{n+1}(B, \pi_n F)$ which is zero if and only if s_n can be extended to a section on the $(n+1)$-skeleton of B. ☐

Lemma 12.12. *There exists a cohomology class $\mathfrak{o}_n(\Sigma) \in H^n(\Sigma; \pi_{n-1} SO(n+1))$ so that $T\Sigma \oplus \epsilon^1$ is trivial if and only if $\mathfrak{o}_n = 0$.*

Proof. Let $\rho : E \to \Sigma$ be the frame bundle of $T\Sigma \oplus \epsilon^1$. Then $T\Sigma \oplus \epsilon^1$ is trivial if and only if ρ has a global section. Let $s_0 : \Sigma^0 \to E$ be any section of ρ on the 0-skeleton of Σ. By Lemma 12.11, the obstruction to extending s_0 to the 1-skeleton Σ^1 lives in $H^1(\Sigma, \pi_0 SO(n+1))$. Since Σ is a homotopy n-sphere, $H^1(\Sigma, \pi_0 SO(n+1)) = 0$. We can therefore extend s_0 to Σ^1. Since $H^i(\Sigma, \pi_{i-1} SO(n+1)) = 0$ for $0 < i < n$, we can extend s_0 to a section s_{n-1} on the $(n-1)$-skeleton. The obstruction to extending s_{n-1} to a global section, and therefore to making $T\Sigma \oplus \epsilon^1$ trivial, lives in $H^n(\Sigma; \pi_{n-1} SO(n+1))$. ☐

Since Pontryagin classes are characteristic classes of the complexification of the bundle, we will first translate our problem over into the complex world. Let $q : SO(n+1) \to U(n+1)$ inducing the complexification map $\bar{q} : BSO(n+1) \to BU(n+1)$. Let ξ denote the frame bundle of $T\Sigma \oplus \epsilon^1$ and ξ^C its complexification. In other words, if $f : \Sigma \to BSO(n+1)$ classifies $T\Sigma \oplus \epsilon^1$, then ξ^C is the principal $U(n+1)$-bundle classified by $\bar{q} \circ f$. The Pontryagin class $p_k(\xi)$ is equal to the $\pm c_{2k}(\xi^C)$. As for our obstruction class,

$$q_* \mathfrak{o}_n(\Sigma) = q_*([f]) = [\bar{q} \circ f]$$

is the obstruction to ξ^C being trivial.

Lemma 12.13. *Let $h : U(n+1) \to U(n+1)/U(2k-1)$ be projection. The Chern class $c_{2k}(M)$ is equal to $h_*([\bar{q} \circ f])$ and hence $h_* q_* \mathfrak{o}_n(\Sigma) = \pm p_k(M)$.*

Here we are interpreting

$$c_{2k}(M) \in H^n(M, \mathbb{Z}) \cong H^n(M, \pi_{n-1}(U(n+1)/U(2k-1))) \cong \pi_{n-1}(U(n+1)/U(2k-1))$$

since $\pi_{n-1}(U(n+1)/U(2k-1)) \cong \mathbb{Z}$ by Bott periodicity and we transition from cohomology to homotopy groups using the fact that Σ is a homotopy sphere.

Remark. Recalling our third description of obstruction classes,

$$[\bar{q} \circ f] \in \pi_{n-1}(U(n+1)) \cong H^n(\Sigma, \pi_{n-1}(U(n+1)))$$

is the obstruction to extending a section of ξ^C over the $(n-1)$-skeleton of M to all of M. The element

$$h_*([\bar{q} \circ f]) \in \pi_{n-1}(U(n+1)/U(2k-1)) = H^n(\Sigma, \pi_{n-1}(U(n+1)/U(2k-1)))$$

is the obstruction to extending a partial framing of $(n+1)-(2k-1) = 2k+1$ frames from the $(n-1)$-skeleton of M to the whole space.

Proof. Since Chern classes and our obstruction classes are both natural, it suffices to show that $c_{2k}(\gamma^n) = h_*\mathfrak{o}_k(\gamma^n)$ where γ^n is the universal bundle on $BU(n)$. The elements $c_{2k}(\gamma^n)$ and $h_*\mathfrak{o}_k(\gamma^n)$ live in the group

$$H^n(BU(n), \pi_{n-1}U(n+1)/U(2k-1)) \cong H^n(BU(n), \mathbb{Z})$$

The cohomology ring $H^*(BU(n), \mathbb{Z})$ is a polynomial ring $\mathbb{Z}[c_1, \dots, c_n]$. Thus $h_*\mathfrak{o}_k(\gamma^n)$ looks like $f'(c_1, \dots, c_{2k-1}) + \lambda c_{2k}$ where f' is some polynomial and $\lambda \in \mathbb{Z}$. We will show $f' = 0$ and then that $\lambda = 1$.

Consider the bundle $\gamma^{2k-1} \oplus \epsilon^1$ over $BU(2k-1)$. Since this bundle has a section, the obstruction $\mathfrak{o}_n(\gamma^{2k-1} \oplus \epsilon^1)$ is trivial. Thus

$$0 = h_*\mathfrak{o}_n(\gamma^{2k-1} \oplus \epsilon^1) = f'(c_1(\gamma^{2k-1}), \dots, c_{2k-1}(\gamma^{2k-1}) + \lambda \cdot 0$$

since the first $2k-1$ Chern classes of γ^{2k-1} are linearly independent, we must have $f' = 0$.

To check that $\lambda = 1$, consider the restriction γ_1^k of γ^k to $Gr_k(\mathbb{C}^{k+1}) \subset Gr_k(\mathbb{C}^\infty) = BU(n)$. The Grassmannian $Gr_k(\mathbb{C}^{k+1})$ is the same as the projective space $\mathbb{CP}^k$. Thinking of $\mathbb{CP}^k$ as a quotient of S^{2k}, we can describe the fiber of $\gamma_1^k \oplus \epsilon^{n-k}$ over a pair $\{u, -u\}$ as the orthogonal points

$$(\gamma_1^n)|_{\{u,-u\}} = \{v \in \mathbb{C}^{2k+1} : u \cdot v = 0\}$$

We can build a cross-section of $\gamma_1^k \oplus \epsilon^{n-k}$ over $\mathbb{CP}^k \setminus \{u_0, -u_0\}$ by mapping

$$\{u, -u\} \mapsto (u_0 - (u_0 \cdot u)u, 1, 1, \dots, 1)$$

with $n-k$ ones corresponding to the rank $n-k$ trivial bundle. We therefore have a cross-section of γ_1^k over the $(k-1)$-skeleton of $\mathbb{CP}^k$. The obstruction to extending this cross-section to all of $\mathbb{CP}^k$ is ± 1. $\square$

12.4. **Hirzebruch Signature Theorem.** We prove the following

Theorem 12.14. *For any closed oriented* $4n$*-manifold* M *we have* $\operatorname{sign}(M) = \langle L(M), [M] \rangle$.

Definition. The L-polynomial is $L(u) = \frac{u}{\tanh(u)}$. The L-class of M^{4n} is

$$L(M) := L_n(p_1, \dots, p_n) = [L(u_1) \cdots L(u_n)]_{4k}$$

i.e., the $4k$-dimensional part of the product of the L-polynomials $L(u_1), \dots, L(u_n)$. Here $p_i = p_i(M)$ is the ith Pontryagin class of M identified as the ith symmetric polynomial in the variables $u_1, \dots, u_n$.

Define the signature of manifolds with dimension not divisible by four to be zero.

Since $\mathbb{Z} \to \mathbb{Z} \otimes \mathbb{Q}$ is injective, it suffices to consider everything over $\mathbb{Q}$. We will show sign and $\langle L(M), [M] \rangle$ define ring maps $\Omega_*^{SO} \otimes \mathbb{Q} \to \mathbb{Q}$. Since $\Omega_*^{SO} \otimes \mathbb{Q}$ is a polynomial ring on even dimensional complex projective spaces, its suffices to show sign and $\langle L(M), [M] \rangle$ agree on these generators. We will then compute

$$\operatorname{sign}(\mathbb{CP}^{2i}) = 1 = \langle L(\mathbb{CP}^{2i}), [\mathbb{CP}^{2i}] \rangle$$

This will complete the proof.

Lemma 12.15. *The map* $\mathrm{sign} : \Omega_*^{SO} \otimes \mathbb{Q} \to \mathbb{Q}$ *is a well-defined ring map.*

The proof given here is following [30, pg 220].

Proof. By Lemma 8.2, the map is well-defined. We show that the signature is a ring map. Let M^m, N^n manifolds. We need to show $\mathrm{sign}(M \times N) = \mathrm{sign}(M) \cdot \mathrm{sign}(N)$. If $m + n$ is not divisible by four, then either m or n is also not divisible by four. Say $m \not\equiv 0 \mod 4$. Then

$$0 = \mathrm{sign}(M \times N) = 0 \cdot \mathrm{sign}(N) = \mathrm{sign}(M) \cdot \mathrm{sign}(N)$$

Say $n = 4k$. By the Künneth theorem we have

$$H^{2k}(M \times N) \cong \sum_{s=0}^{2k} H^s M \otimes H^{2k-s} N$$

Outside the middle dimension, things come in pairs

$$(H^s M \otimes H^{2k-s} N) \oplus (H^{m-s} M \otimes H^{n-(2k-s)} N)$$

for $s < m/2$. Note that $n - (2k - s) = 2k + s - m$. If one of m and n is even, then both are and we also get a term in the middle dimension: $H^{m/2} M \otimes H^{n/2} N$. This is where the signatures of M and N are coming from. Under this decomposition (into the middle term and all the paired non-middle terms), the pairing is orthogonal on distinct parts and nondegenerate when restricted to each part. We can therefore compute the signature of $M \times N$ by computing the signature of each part of our decomposition.

First, we show that the signature on non-middle terms

$$H^{2k}(M \times N) \cong \sum_{s=0}^{2k} H^s M \otimes H^{2k-s} N$$

is zero. Fix $s < m/2$. Take bases $\{x_i\}$ and $\{y_i\}$ for $H^s M$ and $H^{2k-s} N$, respectively. Let $\{x_i^*\}$ and $\{y_i^*\}$ be the dual bases for $H^{m-s} M$ and $H^{n-(2k-s)} N$. Then $\{x_i \otimes y_j, x_p^* \otimes y_q^*\}$ is a basis for the sth non-middle term. Everything pairs to zero except dual things which pair as

$$(x_i \otimes y_j) \cdot (x_i^* \otimes y_j^*) = (x_i^* \otimes y_j^*) \cdot (x_i \otimes y_j) = \pm 1$$

The corresponding matrix looks like $\begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix}$ which has signature zero. Thus if m and n are odd, $\mathrm{sign}(M \times N) = 0$.

If m and n are even then the signature of $M \times N$ equals the signature of the pairing on the middle terms $H^{m/2} M \otimes H^{n/2} N$. If m and n are both divisible by 4 then take diagonalizing bases for $H^{m/2} M$ and $H^{n/2}$. This gives a diagonalizing basis for $H^{m/2} M \otimes H^{n/2} N$ with signature $\mathrm{sign}(M) \cdot \mathrm{sign}(N)$. Otherwise both n, m are equivalent to 2 mod 4. In this case, we have $\mathrm{sign}(M) = 0 = \mathrm{sign}(N)$. Choose symplectic bases $\{a_i, b_i\}$ and $\{c_i, d_i\}$ for $H^{m/2} M$ and $H^{n/2} N$, respectively. Then the matrices for the pairings on $H^{m/2} M$ and on $H^{n/2} N$ both look like direct sums of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. The matrix for the pairing on $H^{m/2} M \otimes H^{n/2} N$ then looks like

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

which has signature zero. $\square$

Lemma 12.16. *The L-genus $\Omega_*^{SO} \otimes \mathbb{Q} \to \mathbb{Q}$ defined by $M \mapsto \langle L(M), [M]\rangle$ is a well-defined ring map.*

Proof. Let $M = \partial W$ be a nullbordant $4n$-manifold. The boundary map $\partial : H_{4n+1}(W, M) \to H_{4n}(M)$ maps $\partial([W]) = [M]$. Let $\delta : H^{4n}(M; \mathbb{Q}) \to H^{4n+1}(W, M; \mathbb{Q})$ be the dual map. Then

$$\langle L(M), [M]\rangle = \langle L(M), \partial([W])\rangle = \langle \delta L(M), \partial([W])\rangle$$

Let $i^* : H^{4n}(W; \mathbb{Q}) \to H^{4n}(M; \mathbb{Q})$ induced by inclusion. Since $TW|_M = TM \oplus \epsilon^1$, the Pontryagin classes of W restricted to M are the Pontryagin classes of M. Now $L(M)$ is a polynomial in the Pontryagin classes $p_1(M), \ldots, p_n(M)$ of M. Thus $L(M)$ is in the image of i^*. Since

$$H^{4n}(W; \mathbb{Q}) \xrightarrow{i^*} H^{4n}(M; \mathbb{Q}) \xrightarrow{\delta} H^{4n+1}(W, M; \mathbb{Q})$$

is exact, we have $\delta L(M) = 0$. Hence $\langle \delta L(M), [W]\rangle = 0$. The L-genus is therefore well-defined. □

The main benefit of working over $\mathbb{Q}$ is that the hard to compute torsion of Ω_*^{SO} vanishes.

Theorem 12.17. *The ring $\Omega_*^{SO} \otimes \mathbb{Q}$ is a polynomial ring on generators $\{\mathbb{CP}^{2i}\}$.*

For a proof see [22, Cor 18.9].
It therefore suffices to consider the case of $\mathbb{CP}^{2i}$.

Lemma 12.18. *The signature of $\mathbb{CP}^{2i}$ is 1,*

$$\mathrm{sign}(\mathbb{CP}^{2i}) = 1 = \langle L(\mathbb{CP}^{2i}), [\mathbb{CP}^{2i}]\rangle.$$

Proof. The ring $H^*(\mathbb{CP}^{2i}, \mathbb{Z})$ is isomorphic to $\mathbb{Z}[x]/x^{2i+1}$ with $|x| = 2$. Thus $H^{2i}(\mathbb{CP}^{2i}, \mathbb{Z}) \cong \mathbb{Z}\{x^i\}$ and

$$x^i \cup x^i = x^{2i} \in H^{4i}(\mathbb{CP}^{2i}, \mathbb{Z}) \cong \mathbb{Z}\{x^{2i}\}$$

so that $\mathrm{sign}(\mathbb{CP}^{2i}) = 1$.

Next we show $\langle L(\mathbb{CP}^{2i}), [\mathbb{CP}^{2i}]\rangle = 1$ for any i. Note that the tangent bundle

$$T\mathbb{CP}^{2i} \oplus \epsilon^1 \cong \mathcal{O}(-1)^{\oplus 2i+1}$$

stably splits as a sum of canonical line bundles.

The only nontrivial Chern class of $\mathcal{O}(-1)$ is $c_1 = x \in H^2(\mathbb{CP}^{2i}, \mathbb{Z}) \cong \mathbb{Z}\{x\}$. We have

$$(1 - p_1) = (1 - c_1)(1 + c_1) = 1 - x^2$$

so that $p_1 = x^{4i}$. Thus the only nonzero Pontryagin class is x^{2i} and $\langle x^{2i}, [\mathbb{CP}^{2i}]\rangle = 1$. Hence

$$\langle L(\mathbb{CP}^{2i}), [\mathbb{CP}^{2i}]\rangle = \langle L(\mathcal{O}(-1)^{\oplus 2i+1}), [\mathbb{CP}^{2i}]\rangle = \langle \Big(\frac{x}{\tanh x}\Big)^{2i+1}, [\mathbb{CP}^{2i}]\rangle$$

is the coefficient on x^n in $\big(\frac{x}{\tanh x}\big)^{i+1}$. By the Cauchy integral formula, this coefficient is equal to the integral

$$\frac{1}{2\pi i}\int_C \frac{1}{y^{i+1}}\Big(\frac{x}{\tanh x}\Big)^{2i+1} dx$$

where C is a small circle around 0. Substituting $z = \tanh x$ we obtain

$$\frac{1}{2\pi i}\int \frac{1}{z^{2i+1}(1 - z^2)} dz = \frac{1}{2\pi i}\int \frac{1}{z^{2i+1}}\Big(\sum_{j=0}^{\infty} z^{2j}\Big) dz$$

Now $\int \frac{z^{2j}}{z^{2i+1}}$ is zero unless $i = j$ in which case the integral is 1. Thus

$$\langle L(\mathbb{CP}^{2i}), [\mathbb{CP}^{2i}]\rangle = 1 = \mathrm{sign}(\mathbb{CP}^{2i})$$

□

This completes the proof of Theorem 12.14.

12.5. **Assumed Results.** We give a brief summary of the results used but not proven here.

12.5.1. *Pontryagin-Thom.* The Pontryagin-Thom theorem states that for each n there is an isomorphism $\Omega_n^{fr} \cong \pi_n^S$. The map $\Omega_n^{fr} \to \pi_n^S$ is defined as follows. Let $i : M \hookrightarrow \mathbb{R}^{n+k}$ be an n-manifold embedded in $\mathbb{R}^{n+k}$ with trivialized normal bundle, $\bar{u} : \mathrm{Norm}(i) \cong M \times \mathbb{R}^k$. By the tubular neighborhood theorem, there is an embedding ϕ of the normal bundle in $\mathbb{R}^{n+k}$ so that

$$\begin{tikzcd} M \arrow[r, "i"] & \mathbb{R}^{n+k} \\ \mathrm{Norm}(i) \arrow[u] \arrow[ur, "\phi"'] & \end{tikzcd}$$

commutes. Define a collapse map between the one-point compactifications $(\mathbb{R}^{n+k})^+ \to \mathrm{Norm}(i)^+$ by

$$x \mapsto \begin{cases} \phi^{-1}(x) & x \in \mathrm{Im}\,\phi \\ + & \text{otherwise} \end{cases}$$

The image of M under $\Omega_n^{fr} \to \pi_n^S$ is then defined to be the homotopy class of the map

$$S^{n+k} = (\mathbb{R}^{n+k})^+ \to \mathrm{Norm}(i)^+ \xrightarrow{\bar{u}} (M \times \mathbb{R}^k)^+ \to (\mathbb{R}^k)^+ = S^k$$

The inverse map $\pi_n^S \to \Omega_n^{fr}$ is defined as follows. Let $\gamma : S^{n+k} \to S^k$ be a representative of some class in π_n^S. Defined the image of $[\gamma]$ under $\pi_n^S \to \Omega_n^{fr}$ to be the n-manifold $\gamma^{-1}\{y\}$ for some regular value y. One can give $\gamma^{-1}\{y\}$ a normal framing. The details of checking that these maps are well-defined and inverses of each other can be found in [31].

12.5.2. *Finiteness of π_r^S.* In his seminal paper [27], Serre proved that the stable homotopy groups of spheres are finite. His proof involves use of the Serre spectral sequences as well as hard computations involving power series. Serre was awarded a Fields Medal in part for this theorem, which is a foundational result in modern algebraic topology, underlying a large number of important results.

12.5.3. *Adams' $J(X)$.* In his paper *On the groups $J(X)$ IV*, Adams computed the image of the J-homomorphism $J : \pi_n SO \to \pi_n^S$. When $r \equiv 0, 1 \mod 8$, he showed that $J : \pi_r(SO) = \mathbb{Z}/2 \to \pi_r^S$ is injective. For $r \equiv 3, 7 \mod 8$, Adams showed that $\mathrm{Im}\, J$ is a cyclic group of order $\frac{B_s}{4s}$ where $r = 4s-1$ and B_s is the sth Bernoulli number. The case $r \equiv 7 \mod 8$ requires the Adams conjecture which was proven by Quillen in [24].

Adams defines an e-invariant on a subgroup of π_n^S that detects the image of J. The e-invariant lands in certain Ext groups in KO-theory which Adams is able to compute. A discussion of the relationship between Adams' e-invariant and parts of the Kervaire-Milnor sequence can be found in [8]. Adams' proof relies on major theorems including the existence of a Thom isomorphism for spin vector bundles in KO as well as difficult computations of "cannibalistic classes" from Adams' papers $J(X)I - III$ and *Vector Fields on Spheres.*

12.5.4. *Computation of $\pi_{2m}U(m)$.* In [4], Bott computes $\pi_{2m}U(m) \cong \mathbb{Z}/m!$. Bott develops a general theory for computing the Hopf algebra $H_*\Omega G$ for G a compact, connected, Lie group. He then applies his theory to the case $G = SU(m)$ to prove Bott periodicity. The Hopf algebra $H_*\Omega SU(m)$ is a polynomial ring and Bott makes use of this nice structure, together with his periodicity theorem, to prove that $\pi_{2m}U(m) \cong \mathbb{Z}/m!$.

12.5.5. *Kervaire Invariant One.* In [12] Kervaire introduced the Kervaire invariant. He used the invariant to prove that certain 10-dimensional manifolds do not admit any smooth structure. In [13], Kervaire and Milnor use the Kervaire invariant as an analogue of the signature in dimensions $4k+2$. We saw this in §8.2. In [6], Browder extended Kervaire's result for 10-manifolds to the theorem that there does not exist an n-manifold with Kervaire invariant one for $n \neq 2^k - 2$. Additionally, Browder showed that a framed manifold of dimension $2k+2$ with Kervaire invariant 1 exists if and only if a certain element in the Adams spectral sequence survives. Using Browder's translation of the Kervaire invariant problem into a problem in stable homotopy theory as well as some equivariant homotopy theory, Hill, Hopkins, and Ravenel in [11] proved that no Kervarie invariant one (2^k-2)-manifold exists for $k \geq 8$. Thus Kervaire invariant one manifolds can only exist in dimensions $2, 6, 14, 30, 62$, and 126.

Framed manifolds with Kervaire invariant one have been constructed in dimensions 2, 6, 14,30, and 62. By the results in §8.2, the group $bP_{2k} = \mathbb{Z}/2$ unless there exists $M \in \Omega_{2k}^{alm}$ with Kervaire invariant $c(M) = 1$, in which case $bP_{2k} = 0$. By Corollary 8.8, we have $bP_{2k} = 0$ for $k = 1, 3, 7$. Thus a framed manifold with Kervaire invariant one exists in dimensions 2, 6, and 14. In [17], Mahowald and Tangora showed that there exists a 30-dimensional manifold with Kervaire invariant one. In [2], Barratt, Jones, and Mahowald constructed a 62-dimensional manifold with Kervaire invariant one. The question in dimension 126 is still open.

12.5.6. *Computation of π_r^S.* The computations of π_r^S used here are summarized in the table at the beginning of §10. All results used here can be determined using Serre's method. One computes $\pi_{n+r}S^n$ by using the Serre spectral sequence to move up the Postnikov tower. The computations are done one prime at a time. A walk-through of some of these computations can be found in [23]

12.5.7. *The Lie group $SU(3)$ as a framed manifold.* Let $\mathcal{L}$ denote the framing of the stable normal bundle of $SU(3)$ using the left-invariant trivialization of $TSU(3)$. Then $[SU(3);\mathcal{L}] \in \Omega_8^{fr}$ represents the class of a framed manifold. Under the Pontryagin-Thom isomorphism, there is a corresponding element of π_8^S. This element was identified by Smith, Steer, and Wood in [28], [29], and [33], respectively. The proof involves computing Hopf and e-invariants. In [29] and [33] various other framings of $SU(3)$ obtained by twisting by a representation of $SU(3)$ are studied. In fact, they show that every element in π_8^S can be obtained by some framing of $SU(3)$.

12.5.8. *Computation of $\Omega_*^{SO} \otimes \mathbb{Q}$.* One can show that Ω_*^{SO} is isomorphic, as a ring, to $\pi_* MSO$. Here MSO is the Thom spectrum of SO. The Thom isomorphism implies that in homology, MSO and BSO look the same, $H_*(MSO;\mathbb{Q}) \cong H_*(BSO, \mathbb{Q})$. One then uses the rational Hurewicz theorem and some results of Serre's mod $\mathcal{C}$ theory to show that $\pi_* MSO \otimes \mathbb{Q} \cong H_*(MSO;\mathbb{Q})$. Details can be found in [22, Cor 18.9].

References

[1] Adams, J. F. *On the groups J(X).* IV. Topology **5** (1966), 21-71.

[2] Barratt, M.G.; Jones, J.D.S.; Mahowald, M.E. *Relations amongst Toda brackets and the Kervaire invariant in dimension 62.* J. London Math. Soc. (2) **30** (1984), no. 3, 533-550.

[3] Bott, Raoul. *The stable homotopy of the classical groups.* Ann. of Math. (2) **70** (1959) 313-337.

[4] Bott, Raoul. *The space of loops on a Lie group.* Michigan Math. J. **5** (1958) 35-61.

[5] Bott, Raoul.; Milnor, J. *On the parallelizability of the spheres.* Bull. Amer. Math. Soc. **64** (1958), 87-89.

[6] Browder, W. *The Kervaire invariant of framed manifolds and its generalization.* Ann. of Math. (2) **90** (1969) 157-186.

[7] Browder, W. *Surgery on Simple-Connected Manifolds.* Springer-Verlag, New York, (1972).

[8] Brumfiel, G. *On the homotopy groups of BPL and PL/O.* Ann. of Math. (2) **88** (1968), 291-311.

[9] Brumfiel, G. *The homotopy groups of BPL and PL/O. III.* Michigan Math. J. **17** (1970), 217-224.

[10] Frank, G. *An invariant of almost closed manifolds.*

[11] Hill, M. A.; Hopkins, M. J.; Ravenel, D. C. *On the nonexistence of elements of Kervaire invariant one.* Ann. of Math. (2) **184** (2016), no. 1, 1-262.

[12] Kervaire, Michel A. *A manifold which does not admit any differentiable structure.* Comment. Math. Helv. **34** (1960), 257-270.

[13] Kervaire, Michel A.; Milnor, John W. *Groups of homotopy spheres.* I. Ann. of Math. (2) **77** (1963), 504-537.

[14] Lang, Serge. *Algebra.* Revised third edition. Graduate Texts in Mathematics, 211. Springer-Verlag, New York, 2002. xvi+914 pp.

[15] Levine, J. P. *Lectures on groups of homotopy spheres.* Algebraic and geometric topology (New Brunswick, N.J., 1983), 62-95, Lecture Notes in Math., 1126, Springer, Berlin, (1985).

[16] Lück, Wolfgang. *A basic introduction to surgery theory.* Topology of high-dimensional manifolds, No. 1, 2 (Trieste, 2001), 1-224, ICTP Lect. Notes, 9, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2002.

[17] Mahowald, M.; Tangora, M. *Some differentials in the Adams spectral sequence.* Topology **6** (1967) 349-369.

[18] Milnor, John. *Differentiable structures on spheres.* Amer. J. Math. **81** (1959) 962-972.

[19] Milnor, John. *A procedure for killing homotopy groups of differentiable manifolds.* Proc. Sympos. Pure Math., Vol. III pp. 39-55 American Mathematical Society, Providence, R.I., (1961).

[20] Milnor, John. *Morse theory.* Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51 Princeton University Press, Princeton, N.J. 1963 vi+153 pp.

[21] Milnor, John. *On manifolds homeomorphic to the 7-sphere.* Ann. of Math. (2) **64** (1956), 399-405.

[22] Milnor, John W.; Stasheff, James D. *Characteristic classes.* Annals of Mathematics Studies, No. 76. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974. vii+331 pp.

[23] Mosher, Robert E.; Tangora, Martin C. *Cohomology operations and applications in homotopy theory.* Harper and Row, Publishers, New York-London, (1968) x+214 pp.

[24] Quillen, Daniel. *The Adams conjecture.* Topology **10** (1971), 67-80.

[25] Seifert, Herbert; Threlfall, William. *A Textbook of Topology*, (trans. M. A. Goldman), Pure and Applied Mathematics **89** Academic Press, New York, 1980.

[26] Serre, Jean-Pierre. *A course in arithmetic.* Translated from the French. Graduate Texts in Mathematics, No. 7. Springer-Verlag, New York-Heidelberg, (1973). viii+115 pp.

[27] Serre, Jean-Pierre. *Homologie singulière des espaces fibrés.* Applications. Ann. of Math. (2) **54**, (1951). 425-505.

[28] Smith, Larry. *Framings of sphere bundles, plumbing pairing, and simple Lie groups.* Topology **13** (1974), 401-415.

[29] Steer, Brian. *Orbits and the homotopy class of a compactification of a classical map.* Topology **15** (1976), no. 4, 383-393.

[30] Stong, Robert E. *Notes on cobordism theory.* Mathematical notes Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo (1968).

[31] Thom, René. *Quelques propriétés globales des variétés différentiables.* Comment. Math. Helv. **28**, (1954). 17-86.

[32] Whitney, Hassler. *Differentiable manifolds.* Ann. of Math. (2) **37** (1936), no. 3, 645-680.

[33] Wood, R. M. W. *Framing the exceptional Lie group G_2.* Topology **15** (1976), no. 4, 303-320.